# Algorithm design in Perfect Graphs

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# Graph Vertex Colouring

- A very practical problem
- Planar Graphs
  - V-E+F=2
  - Can be used to show that E is at most 3n-6- so a planar graph is always 7 colorable. We know 4 is correct.
  - NP-hard to distinguish between 3 colorable and 4 colorable graphs
- What about 2 colorable graphs?

# More 2 colorings

- What about more an a cycle of 5 vertices?
  - It needs 3 colors – parity argument
  - But no 3 mutually adjacent vertices – a 3 clique
- Can we construct a graph that has no 4 clique, but needs 4 colours?
  - Groetsch graph.
- Actually possible to construct clique size 2, chromatic number arbitrary graph
  - 17 year old Laszlo lovasz

# Register allocation and interval coloring

- Registers are colours
- Vertices are variable names
- Variable names have scope
- Scope has a nested structure
- How many colours are required?
- View nested scope as an edge
- Number of colours is at least the size of the maximum clique
- Actually, and easily it is seen as sufficient.

# What is it to be Perfect?

- Introduced by Claude Berge in early 1960s
- Coloring number and clique number are one and the same for all induced subgraphs of a Graph
- Note that the coloring number is at least the clique number
- Are they even unequal? - Odd cycles!!!
- To be perfect, induced subgraphs cannot be odd cycles

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# Exercise in Coloring

- For any given two integers, o and c, does there exist a graph whose coloring number is c and clique number is o.
- For o=2 and c=3, answer is obviously yes.
- Construct a graph for o=2 and c=4.
- Answered by Lovasz for arbitrary values of o and c.
- Check text on Graph Theory by Bondy and Murty.

# Perfect Questions

- Is a given graph Perfect?
- Is there a characterization of perfect graphs?
- Is a graph minimally imperfect?
- Do any hard computational exercises become easy on these graphs?
- Are there interesting sub-classes?

This talk: A survey of the first 4 and a sample of the last question

# Characterizations

- Strong Perfect Graph Theorem

A Graph is perfect if and only if it does not contain a odd cycle or its complement as an induced subgraph- last decade Chudnovsky..

- Conjectured by Berge in 1960

- A forbidden subgraph characterization.
- Conjecture settled after many years of research in the first decade of this century.
- Come up with a verification algorithm?

# Results along the way

- Weak Perfect Graph Theorem [Lovasz, Fulkerson]

A Graph is perfect if and only if its complement is perfect.

Further, G is perfect if and only if for each induced subgraph H, the alpha-omega product is at least the number of vertices in H.

- Consequently, independence number is same as clique cover number for all induced subgraph of a perfect graph.

# Polyhdedral Combinatorics

- Main goal-understanding the geometric structure of a solution space.

Visualize the convex hull and find a system of inequalities that specify exactly the convex hull

- Consider the convex hull of stable set incidence vectors
- Consider the clique inequalities
- G is perfect if and only if the convex hull and clique inequality polytope are identical

# Summary of Survey

- Perfect graphs are motivated by coloring issues.
- Connects combinatorial understanding to polyhedral structure in a very rich and fundamental way

Geometric Algorithms and Combinatorial Optimization – Groetschel, Lovasz, Schrijver

Algorithmic Graph Theory and Perfect Graphs – Golumbic

The Sandwich Theorem – Knuth

# Interval Graphs

- A subclass of perfect graphs
- Motivated by many applications
  - Temporal reasoning issues like register allocation
- Given a set of intervals, consider the natural intersection graph for which there is one vertex per interval and an edge indicates a non-empty intersection
- Examples of interval graphs and non interval graphs

# Interval Graphs are perfect

- Given a graph, find an interval representation
- Visualize the intervals as time intervals
- Color the intervals in increasing order of time
- Reuse a color whenever possible and use a new color greedily
- This proves that interval graphs are perfect.
- Key issues: given a graph, does it have an interval represenation.

# Forbidden subgraphs

- Induced cycles of length more than 3
- Asteroidal triples

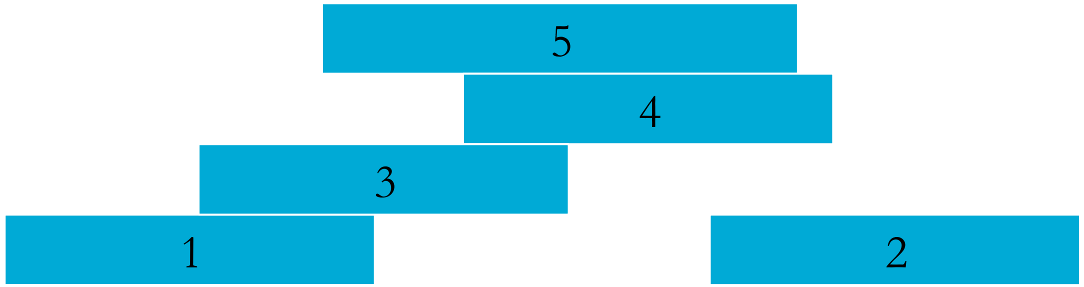
3 vertices x, y, z form an asteroidal triple if for all ordering of them, there is a path from the first to third which avoids the neighbors of the second.

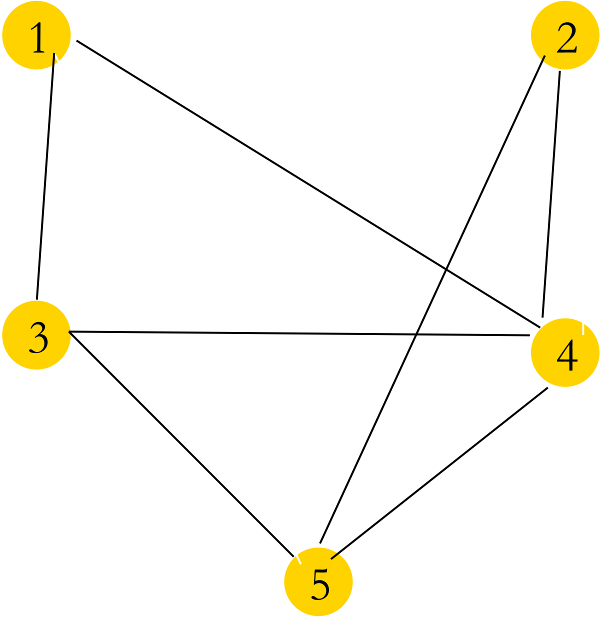
- Gives a polynomial time algorithm
  - Check no four form an induced cycle
  - Check no 3 form an asteroidal triple

# The interval representation

- Graph is an interval graph if and only if its maximal cliques can be linearly ordered such that the set of maximal cliques containing a vertex occur consecutively in the order.
- Note that this consecutive ordering gives the interval representation
  - For each vertex, the interval associated is the interval of indices of maximal cliques that contain it
- Finding the maximal cliques and ordering them!!

# Interval Graphs



# Chordal Graphs

- A Graph in which there is no induced cycle of length four or more.
  - A 4 clique with one edge removed - chordal
  - A 4 cycle with an additional central vertex adjacent to all four - not chordal
- Every interval graph is a chordal graph
- What is the structure of chordal graph?
  - Are they intersection graphs of some meaningful collection of sets?
    - very natural question

# Separators are Cliques

- In chordal graphs minimal vertex separators are cliques
  - structure of minimal separators are very important
  - Also a characterization
- Let X be a minimal u-v separator
  - Assume X is not a clique
  - Because of minimality, for each x in X, in each component (after removal of X), x has a neighbor in the component.
  - Let C1 and C2 be two components

# Why? ..

- Let x1 and x2 be 2 vertices in X, not adjacent
  - Let a1 and a2 be neighbors in C1, and b1 and b2 in C2
  - Then a1 x1 b1 P' b2 x2 a2 P a1 is a cycle
  - From this cycle, we can construct a chordless cycle, contradiction
- The reverse direction
  - If all minimal separators are cliques, no induced cycles.
  - If C is an induced cycle, take x in C and y in C and take any minimal x-y separator containing the neighbors of x in C. Contradiction

# Simplicial Vertices

- A vertex whose neighbor induces a clique
- An incomplete chordal graph has two non-adjacent simplicial vertices!!!
- Proof by induction in the number of vertices
  - a single vertex, is simplicial (Why?)
  - consider an edge, both are
  - consider a path, the degree 1 vertices are (base case)
  - Let X be a minimal separator
    - Consider A + X and B + X

# Since X is a clique..

- apply induction to A+X and B+X
  - they are chordal and smaller.
  - A and B are non-empty
  - take nonadj v_a1, v_a2 in A+X and nonadj v_b1, v_b2 in B+X that are simplicial.
  - at most one of v_a1, v_a2 (v_b1, v_b2) can be in X
  - so we get at least 2 simplicial vertices
- What if A+X is complete, then it is easier.
  - we get a simplicial vertex from A, which is what we want.

# Perfect Simplicial Ordering

- v_1, v_2, ..., v_n is a very special ordering
  - Property: higher numbered numbers of v_i induce a clique in G
- Consequence
  - Color greedily using a simplicial ordering
  - note simplicial ordering can be found in polynomial time.
- And more....

# Finding the maximal cliques

- Based on a structural property of graphs that do not have induced 4 cycles.