t-Spanners for undirected graphs

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Outline

Motivation

State of the art for computing APSP:

	weights	complexity	ref
directed	real	$O(mn + n^2 \lg n)$	[Johnson '77]
directed	integer	$O(mn + n^2 \lg \lg n)$	[Hagerup '00]
undirected	real	$O(mn\alpha(m,n))$	[Pettie-Rama '01]
undirected	integer	$O(mn)$	[Thorup '97]

Given an arbitrary dense graph, find a *sparse subgraph* that approximates all pair distances fairly well.

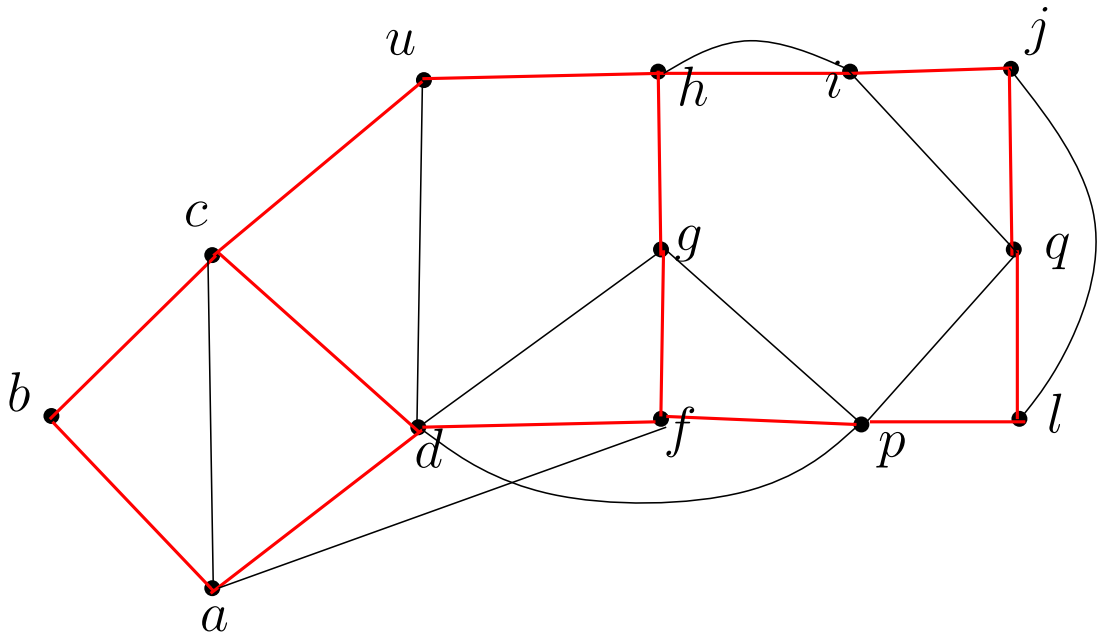
(α, β)-spanner: definition

Given a graph $G(V, E)$, a subgraph $G'(V, E')$ of G is a (α, β)-*spanner* ($\alpha > 1$) of G iff for every $u, v \in V$, $dist_{G'}(u, v) \leq \alpha dist_G(u, v) + \beta$.

* α is the *stretch (dilation) factor* and β is the *surplus* or *additive factor* of G'

t-Spanner: definition

An (α, β)-spanner G' with $\alpha = t(> 1)$ and $\beta = 0$ is known as a *t-spanner* of the given graph G. $\leftarrow$ focus of this talk



2-spanner is in red

Few applications

- APASP in sub-cubic time/sub-quadratic space
- every algorithm that has m-term gets benefitted
- distributed computing
- reconstructing phylogeny trees

t-Spanner: another definition

Given a graph $G(V, E)$, a subgraph $G'(V, E')$ of G is a *t-spanner* ($t > 1$) of G iff for every $u, v \in V$, $dist_{G'}(u, v) \leq t.dist_G(u, v)$.

$\Leftrightarrow$

Given a graph $G(V, E)$, a subgraph $G'(V, E')$ of G is a *t-spanner* ($t > 1$) of G iff for every edge $e(u, v) \in E$, $dist_{G'}(u, v) \leq t.dist_G(u, v)$.

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Ex: A complete graph on n vertices has a 2-spanner of size $n - 1$.

Few lower bounds

- No bipartite graph has a 2-spanner except for the same graph itself.

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- For $t > 2$, it is NP-hard to approximate the smallest size of t-spanner of a graph with $O(2^{(1-\mu)\ln n})$ apprx factor for any $\mu > 0$.

  — not proved in the talk

Erdös girth lower bound conjecture

There are graphs on n vertices whose $(2k-1)$-spanners require $\Omega(n^{1+1/k})$ edges. ([Erdös '63])

proofs exist for $k = 1, 2, 3, 5$ — not proved in the talk

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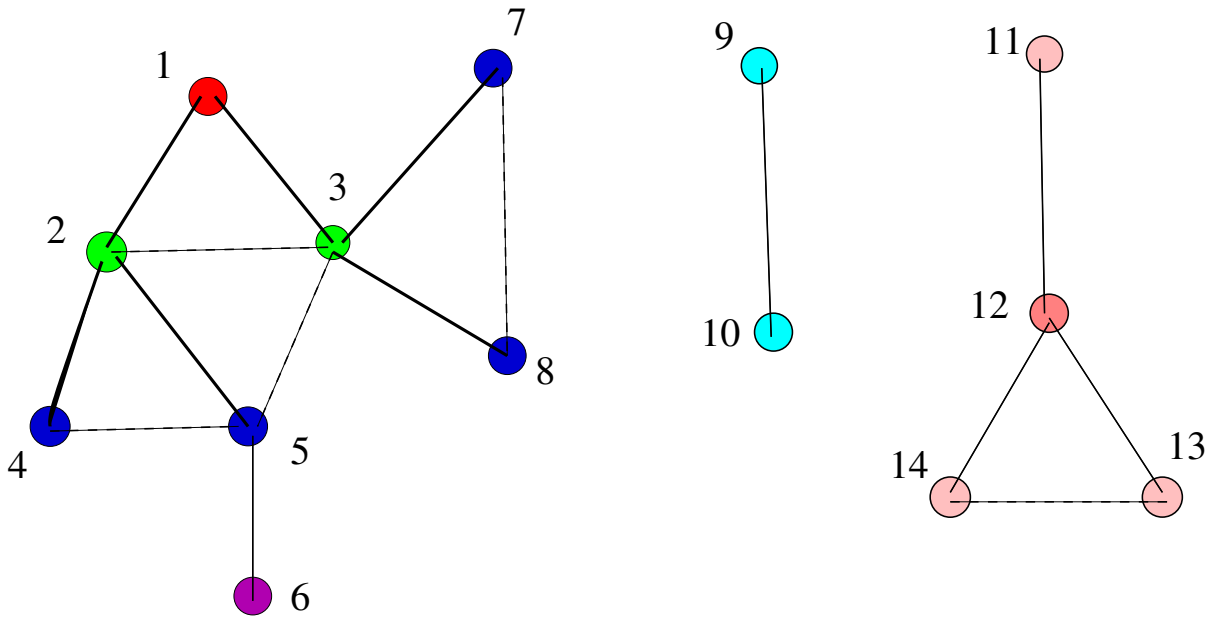
Hence, the objective is to design a linear time algorithm to compute a $(2k-1)$-spanner with $O(n^{1+1/k})$ number of edges for weighted graphs.

Outline

Breadth-first traversal (review)



breadth-first traversal, respectively rooted at $1, 9$ and 11

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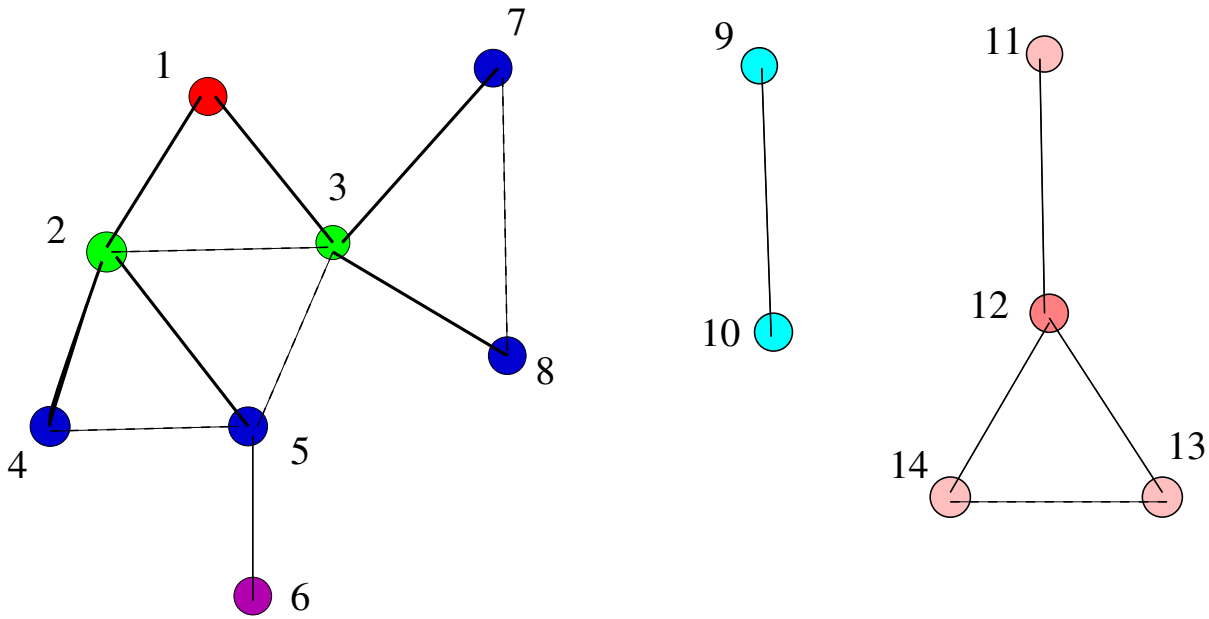


breadth-first traversal, respectively rooted at $1, 9$ and 11

— takes $O(n+m)$ time

For a connected graph G, breath-first traversal tree rooted at any vertex of G is a $O(n)$ spanner of G.

Observation

Partition the vertex set V of G into clusters[1] and introduce as few edges as possible into spanner G' so that

- the distance between any two nodes in a cluster
- as well as the distance between any two nodes from two distinct clusters

are nicely approximated.

[1] cluster means a connected component

Algorithm: high-level view

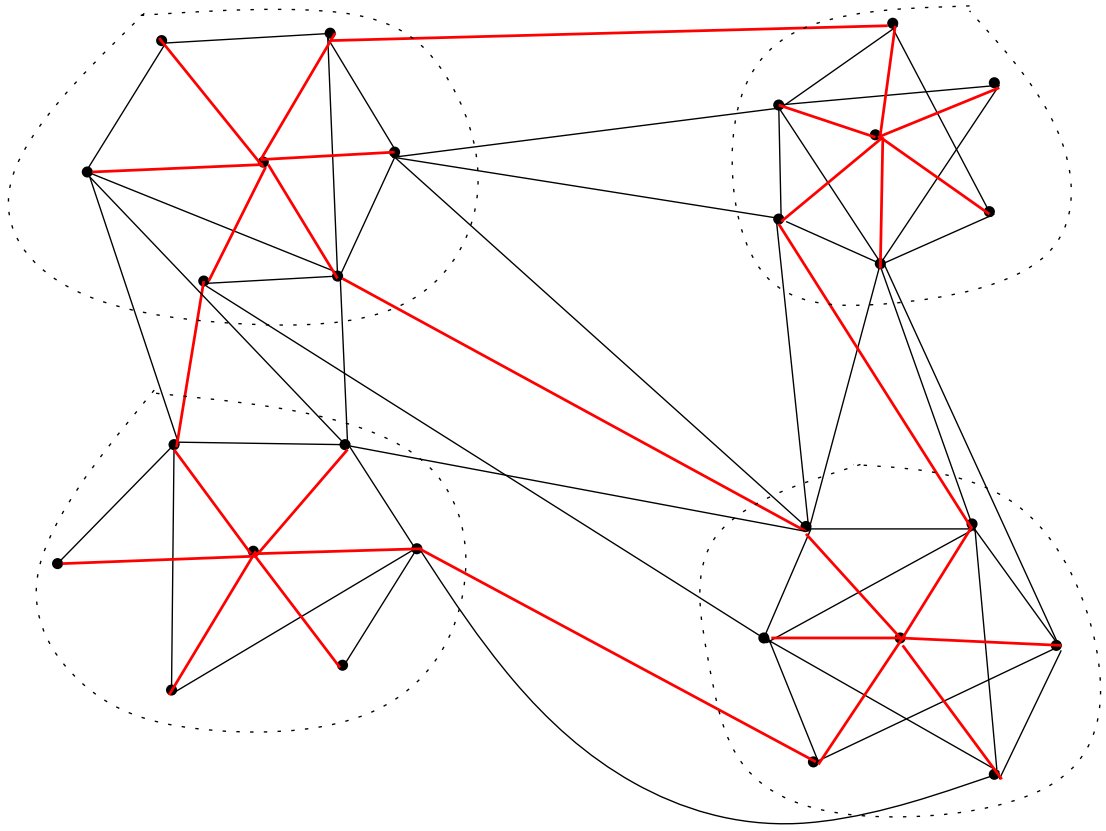
input: undirected unweighted graph $G(V, E)$ and an integer $k \geq 1$

(1) partition V into $\mathcal{T}$ sets such that for every $S_i \in \mathcal{T}$, there exists a vertex c_i such that the distance between c_i and any vertex of S_i in G is $\leq k-1$

(2) Ensure the same in G' by introducing appropriate edges into G': $(\bigcup_i SSSPTree_{c_i}) \cup I'$, wherein set I' comprises of one edge between every two clusters that have at least one edge between them

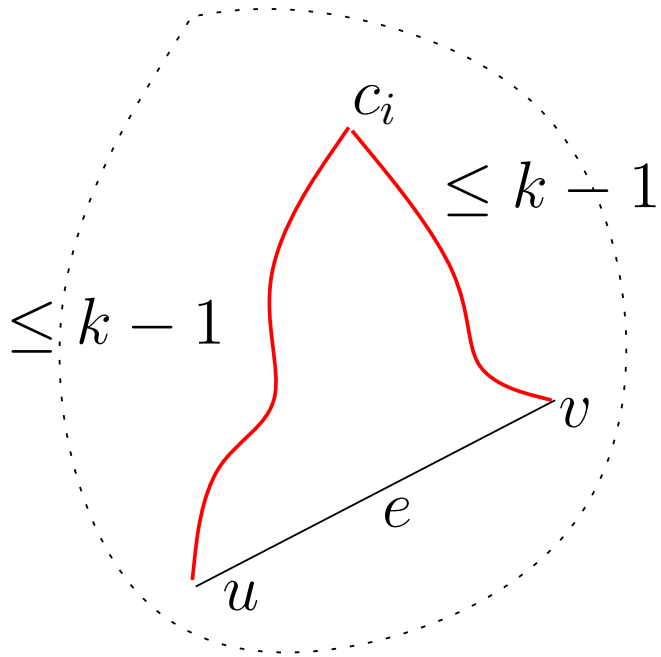
output: $G'(V, E')$ is a $O(k)$-spanner of $G(V, E)$ with $|E'|$ being $O(n^{1+\frac{1}{k}})$ $\leftarrow$ claim

An example



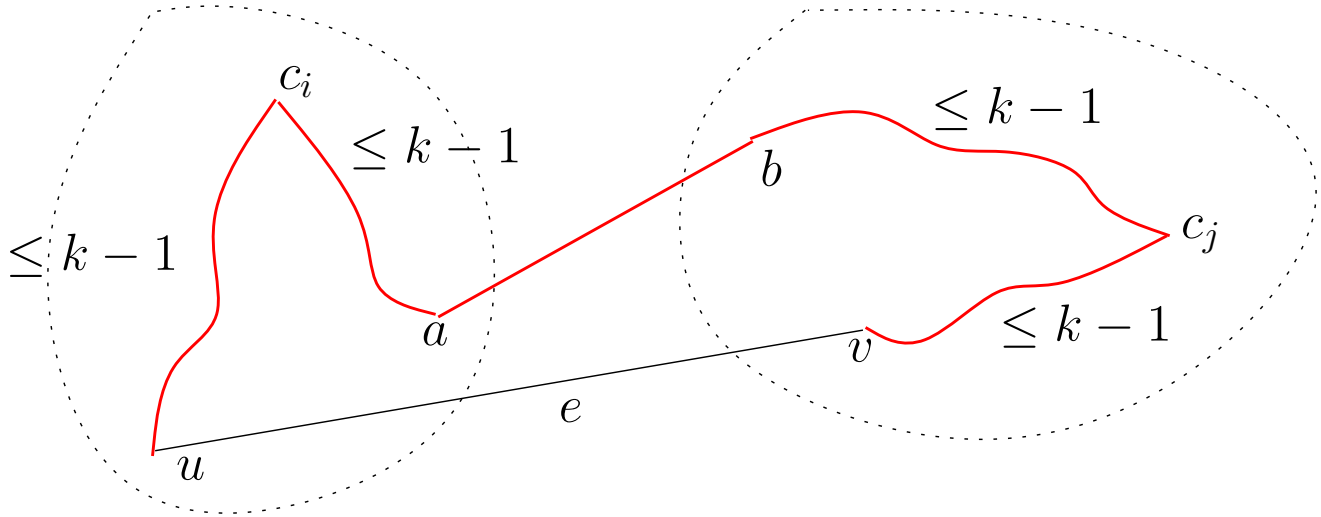
spanner is in red color

G' is a $(4k-3)$-spanner of G: case (i)



both the endpoints of an edge $e \in E$ belong to same cluster

G' is a $(4k-3)$-spanner of G: case (ii)



endpoints of an edge $e \in E$ belong to two distinct clusters

Issue with the above algorithm

How to bound the number of clusters, in turn number of intercluster edges?

Partition V into $\mathcal{T}$

input: undirected unweighted graph $G(V, E)$ and an integer $k \geq 1$

(1) do till every vertex of input graph G belong to a cluster:

- **(i)** for an arbitrary vertex c in the remaining graph, set $S \leftarrow \{c\}$
- **(ii)** while $|S \cup \Gamma(S)| > n^{1/k}|S|$
  - **(a)** include $\Gamma(S)$ to S
- **(iii)** add S to $\mathcal{T}$ and remove all the vertices in S from G

(2) G' comprises of $\bigcup_i SSSP_{c_i} \cup I'$, wherein set I' of edges is formed by choosing one edge between every two clusters that have at least one edge between them

Property (i) of $\mathcal{T}$

For every $S_i \in \mathcal{T}$, $G[S_i]$ is a cluster, and V is indeed paritioned into $\mathcal{T}$.

Property (ii) of $\mathcal{T}$

The cardinality of set I of intercluster edges is upper bounded by $n^{1+\frac{1}{k}}$.

* $|I| \leq \sum_{S_i \in \mathcal{T}} n^{1/k} |S_i| = n^{1+1/k}$

Property (iii) of $\mathcal{T}$

For every $S_i \in \mathcal{T}$, the radius of $G[S_i]$ with respect to a special vertex $c_i \in S$ is upper bounded by $k-1$.

* while building any cluster, number of nodes in it after adding i^{th} layer to it is $> n^{i/k}$

* in any cluster, number of layers added to initial vertex $\leq k-1$

Time complexity

Takes $O(m + n^{1+1/k})$ time to construct G'.

G' is a spanner of interest

$G'(V, E')$ is a $O(k)$-spanner of $G(V, E)$ with $|E'|$ being $O(n^{1+1/k})$.

Lower bounds for unweighted undirected graphs

- For every integer $r \geq 3$, there exist (infinitely many) n-vertex, m-edge graphs with $girth(G) \geq r$ and $m \geq \frac{1}{4}n^{1+1/r}$.

  — not proved in the talk

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Hence, the spanner output by the algorithm is optimal wrt size and (asymptotic) spanning ratio.

Outline

Minimum spanning tree (review)

Objective: Given an undirected weighted connected graph $G(V, E)$, find a tree T that spans all the nodes in V such that T has the minimum weight among all the spanning trees.

MST properties (review)

- *MST cut property*: Assuming all the edge weights are distinct, e is the minimum weighted edge crossing some cut C of $G \Leftrightarrow e \in MST$.

- *MST cycle property*: Assuming all the edge weights are distinct, e is the maximum weighed edge in some cycle O of $G \Leftrightarrow e \notin MST$.

Kruskal's MST algorithm (review)

Start with the spanning forest (SF) comprising vertices of G with no edges included. Consider edges in the order of increasing weight. For an edge $e(u, v)$:

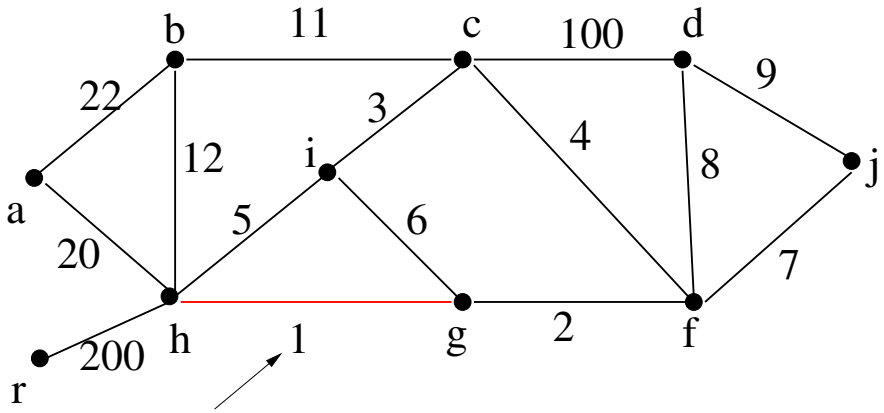
- If there exists a path from u to v in the current SF, do not add e.

  exploits *MST cycle property*

- Otherwise, add e.

  exploits *MST cut property*

Kruskal's algorithm in execution



(i)

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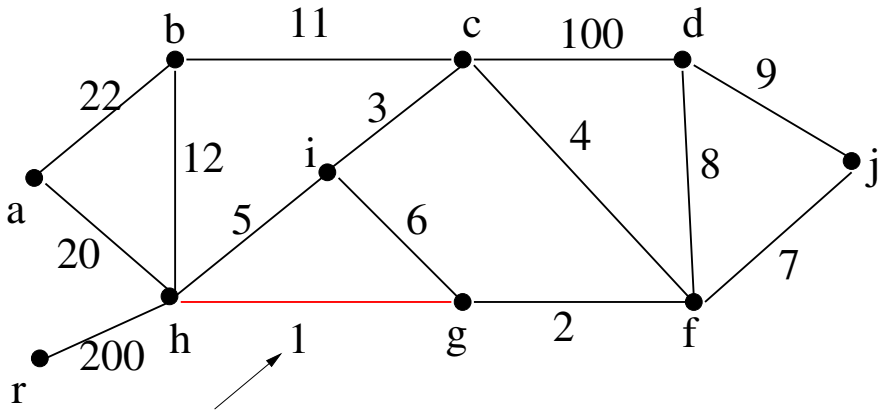

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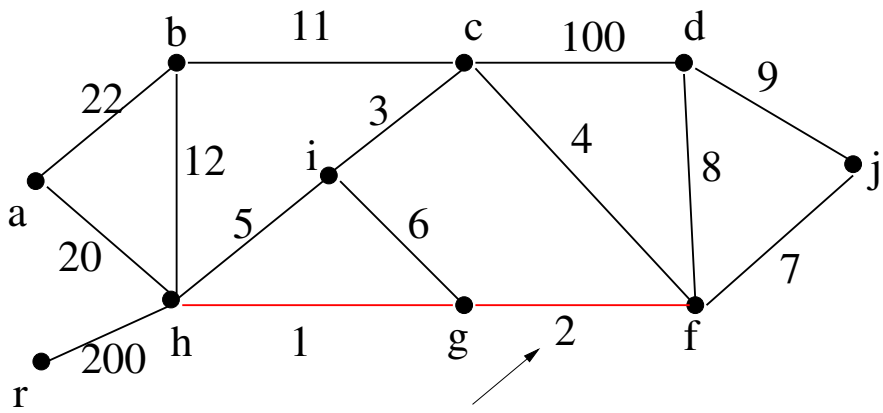

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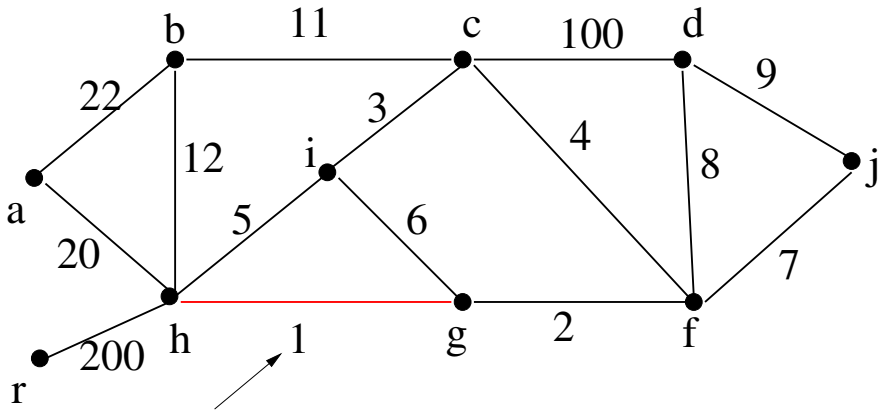

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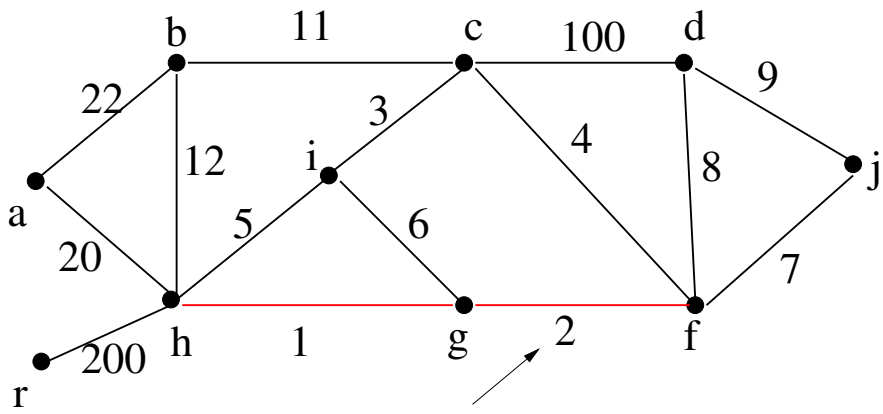

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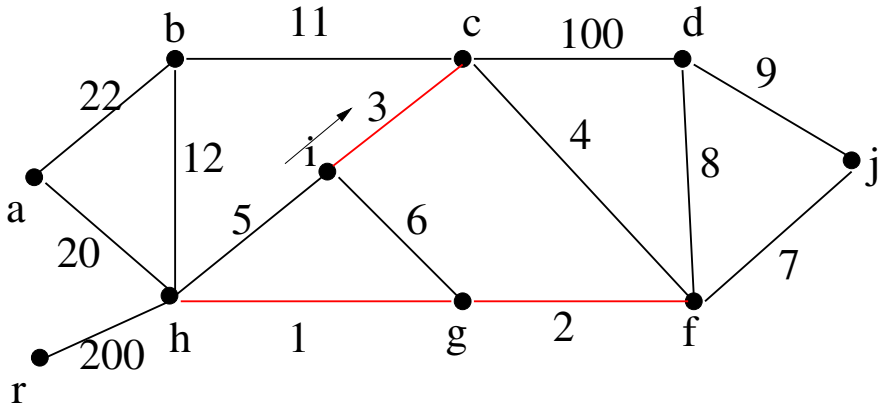

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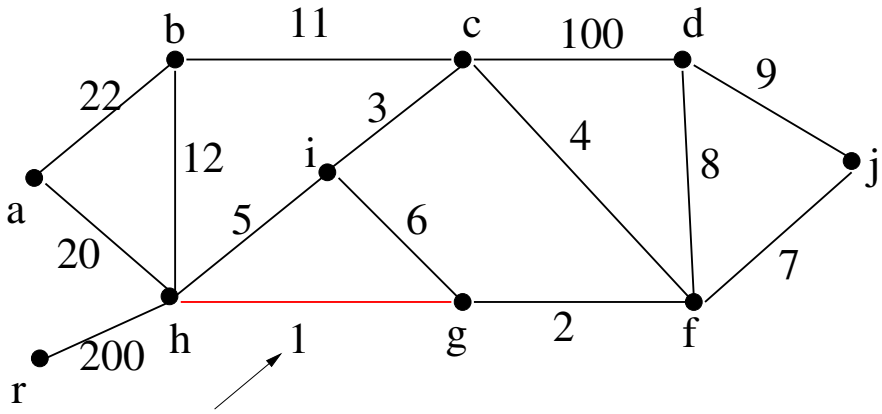

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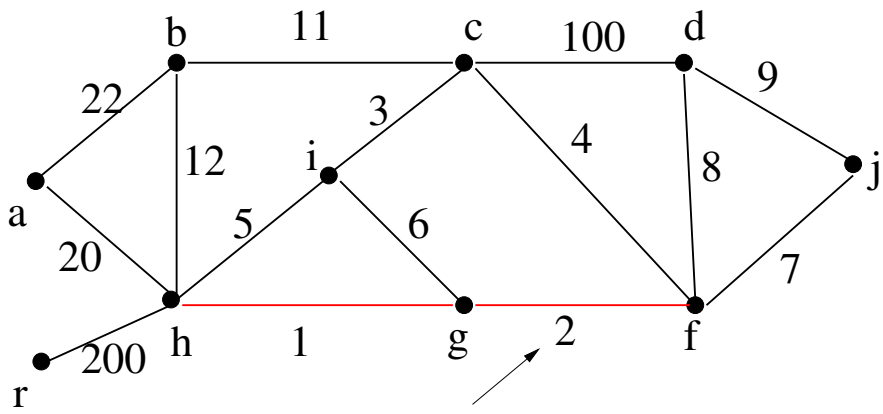

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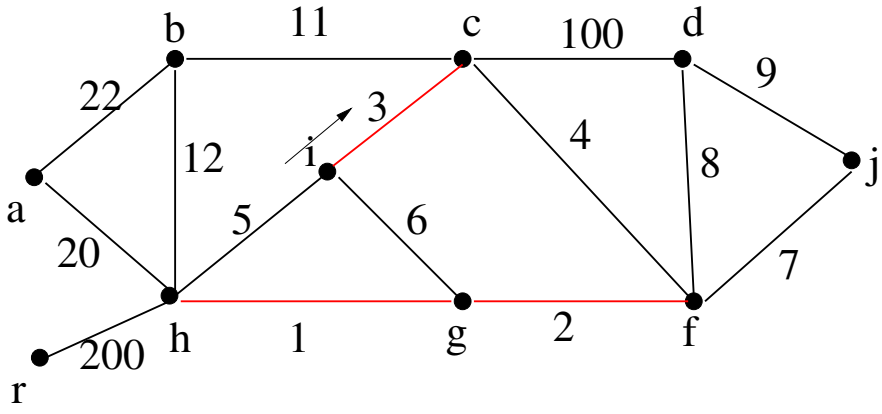

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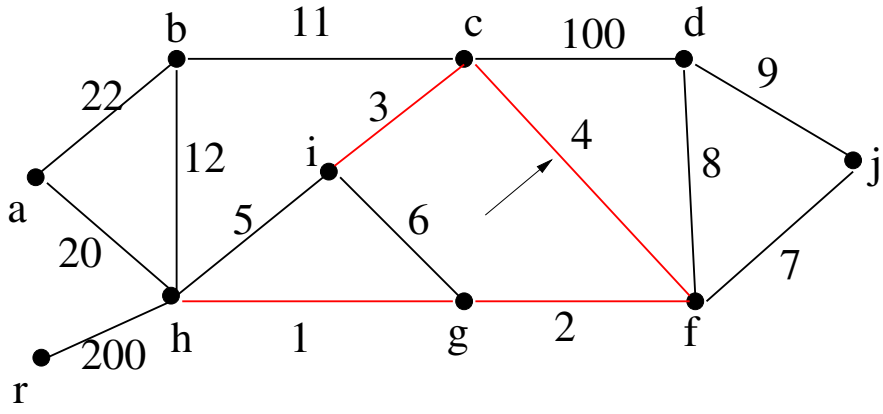

(iv)

Kruskal's algorithm in execution (cont)

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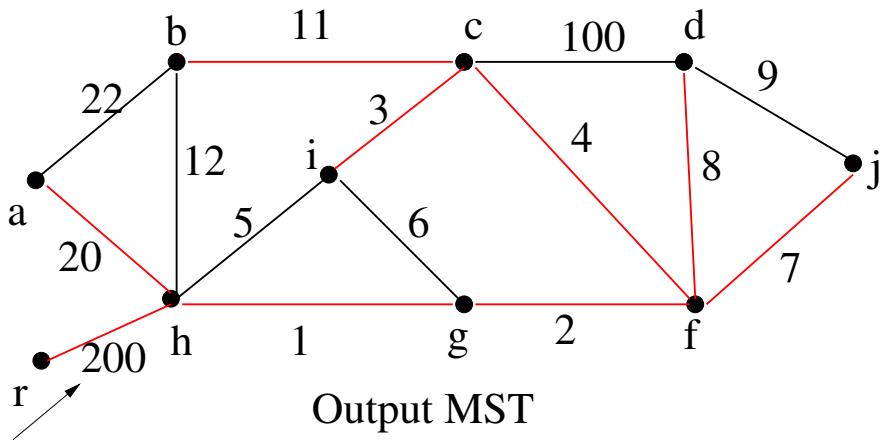

takes $O(|E| \lg |V|)$ time

Few observations from Kruskal's algorithm

- If two components C' and C'' are joined with an edge e during the algorithm, then e is the heaviest weight among the $T_{C'} \cup T_{C''} \cup \{e\}$.

- If the algorithm choses an edge e wherein an endpoint of e incident to a component C', then e is the lightest edge between C' and $V - C'$.

MST T is a $(n-1)$-spanner

let C', C'' be two components in the spanning forest F such that $s' \in C'$ and $s'' \in C''$ just before adding an edge e to F so that C' and C'' are merged

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$$d_T(s', s'')$$

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$$d_T(s', s'')$$
$$\leq (|C'| + |C''| - 1)w_e$$

since e is the heaviest edge in $C' \cup C'' \cup \{e\}$

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$$
\begin{aligned}
& d_T(s', s'') \\
& \leq (|C'| + |C''| - 1) w_e \\
& \qquad \text{since } e \text{ is the heaviest edge in } C' \cup C'' \cup \{e\} \\
& \leq (n-1) w_e
\end{aligned}
$$

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$$d_T(s', s'')$$

$$\leq (|C'| + |C''| - 1)w_e$$

since e is the heaviest edge in $C' \cup C'' \cup \{e\}$

$$\leq (n-1)w_e$$

$$\leq (n-1)d_G(s', s'')$$

since e is the lightest edge between C' and $V - C'$

Lower bound on the stretch of any spanning tree spanner

For a unit-weighted cycle graph, the stretch t can be as bad as $\Omega(n)$.

- hence, Kruskal's algorithm based MST is an optimal spanner with respect to stretch

Disadv with spanning tree spanners: best possible stretch is a function of n

Outline

Objective

Find a t-spanner with

- *size sparsity*: $|E'| < n\lceil n^{\frac{2}{t-1}} \rceil$
- *weight sparsity*: $w(G') < w(MST_G)(1 + \frac{n}{t-1})$

Greedy algorithm

while considering edges in weight nondecreasing order, introduce an edge $e(u,v) \in G$ in G' whenever $dist_{G'}(u,v) > t\,.w(e)$

- every iteration ensures that G' is locally (w.r.t. u and v) a t-spanner (hence, greedy)

Correctness: G' is a t-spanner

For any two vertices u and v, $dist_{G'}(u,v) \leq t\, dist_G(u,v)$.

Correctness: $|E'| < n\lceil n^{\frac{2}{t-1}} \rceil$

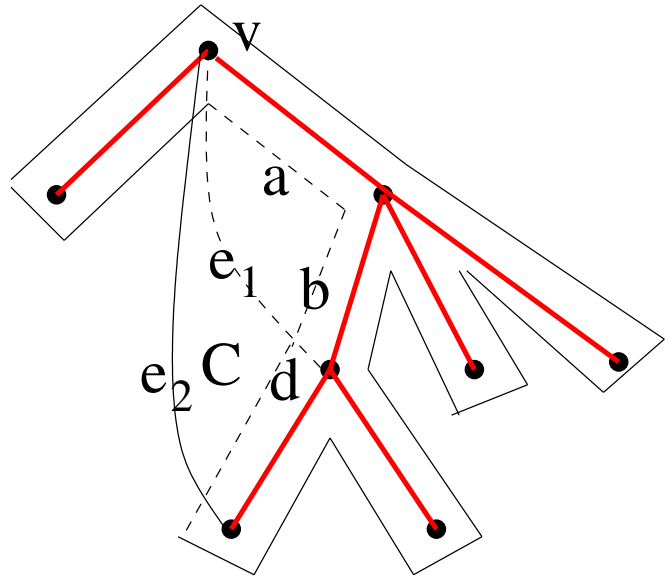
- For any cycle C in G', number of edges of C are $> t+1$; hence, girth size of G' is $> t+1$.
- For any n''-vertex graph G'' with girth $> r$, size of G'' is $< n\lceil n^{\frac{2}{r-2}} \rceil$, leading to the size sparsity.
  - not proved in class

Correctness: $w(G') < w(MST_G)(1 + \frac{n}{t-1})$

MST_G is a subgraph of G'

- compare the algo with the Kruskal's algo for MST: after examining each edge, the number of connected components are same in both; and each component from this algo contains a corresponding component from Kruskal's algo (proof by induction)

Correctness: $w(G') < w(MST_G)(1 + \frac{n}{t-1})$ (cont)



construct skinny polygon P w.r.t. MST_G; for any vertex v, let S_v be the set of edges in G' that have v as one endpoint but do not belong to MST_G; obtain a planar embedding of S_v during the DFT of MST_G with root as v

- for any cycle C in G' and for any edge $e \in C$, $w(C - \{e\}) > t\,w(e)$
- perimeter of P after embedding all the edges in S_v = $2w(MST_G) - (t-1)\sum_{e \in S_v} w(e) > 0$

Analysis

Slightly refined analysis yields a $(2k-1)$-spanner of $O(n^{1+1/k})$ size for any weighted graph.

But takes $O(\min(kn^{2+1/k}, mn^{1+1/k}))$ time.

Outline

Significant $(2k-1)$-spanner algorithms

	size	time	wei
[Althofer et al., '93]	$O(n^{1+1/k})$	$O(mn^{1+1/k})$	w[2]
[Halperin, Zwick '96]	$O(n^{1+1/k})$	$O(m)$	u
[Cohen '98]	$O(n^{1+(2+\epsilon)/(2k-1)})$	$O(mn^{(2+\epsilon)/(2k-1)})$ expc	pw
[Thorup, Zwick '05]	$O(n^{1+1/k})$	$O(kmn^{1/k})$ expc	w
[Baswana, Sen '03]	$O(kn^{1+1/k})$	$O(km)$ expc	w

[2] w: weighted; u: unweighted; p: positive weighted

Current research (weighted graphs)

- $(2k-1)$ spanner of size $O(n^{1+1/k})$ in deterministic linear time
- obtaining < 3 stretch in $n^{2+o(1)}$ time
- purely additive spanners of size $o(n^{4/3})$
- pairwise spanners
- fault-tolerant spanners
- minimum-degree spanners
- dynamic spanners
- a combination of the above

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Thanks!