# Role of Analysis and Linear Algebra in Differential Equations 1

*Examples and General Introduction*

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# Outline

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## General Introduction and Issues

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- Various developments in mathematics: Fourier series, integration and integral equations, complex analysis later linear algebra, functional analysis, operator theory, differential geometry, $\cdots\cdots$.

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- Initial and Boundary Value Problems of ODEs
- Solution Concepts
- Requirement of Mathematical Analysis
- Existence, Uniqueness, Continuous dependence.
- Qualitative Analysis: Stability of Trajectories, Phase Plane Analysis
- Methods to Solve ODEs, Numerical and Scientific Computations (an active area of current research)

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- Any other relevant questions?

## Population Model: Linear

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where $r$ denotes the difference between birth rate and death rate.

• Assuming $r$ is constant, the solution is given by

$$y(t) = y_0 e^{r(t-t_0)} \tag{2}$$

where $y(t_0) = y_0$ is the population at time $t_0$.

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- As $t \to \infty, \ y(t) \to \infty$.
- Indeed, the above linear model is found to be accurate when the population is small. But it cannot be a good model as no population, in practice, can go to $\infty$ as and when the population grows big, there will be competition with each other for the limited available space, food etc.

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- This is known as *logistic law of population* of growth. It was introduced by the Dutch mathematical biologist Verhulst in 1837.

## Population Model: Non-linear Logistic; Conti...

- Practically $b$ is small compared to $a$. If $y$ is not too large, then $by^2$ will be negligible compared to $ay$ and the model is like a linear model. If $y$ becomes large, the term $by^2$ will have a considerable influence on the growth of $y$

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- Using separation of variables, show that $y$ satisfies

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- Hence we need to do further analysis.
- Show that $\frac{y}{y_0} > 0$ and $\dfrac{a-by_0}{a-by} > 0$ for $t > t_0$ and then obtain $y$ as

$$y(t) = \frac{ay_0}{by_0 + (a-by_0)e^{-a(t-t_0)}}. \tag{6}$$

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- The estimation of the vital coefficients $a$ and $b$ in a particular population is indeed an important issue which has to be updated in a period of time as they are influenced by other parameters like pollution, sociological trends etc.

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- In a more realistic model, one needs to consider more than one species, their interactions, unforseen issues like epidemic, natural disasters etc., which may lead to more complicated equations.

# Population Model: Non-linear Logistic; Conti...

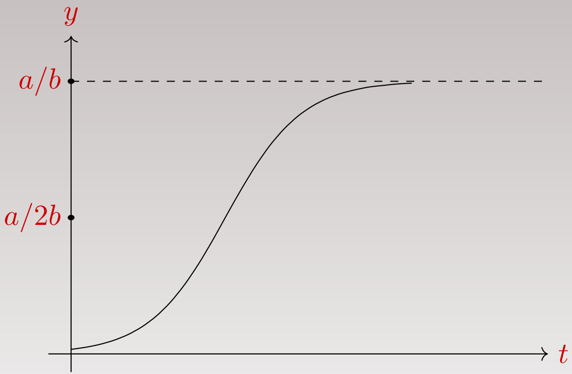


Figure: Logistic map.

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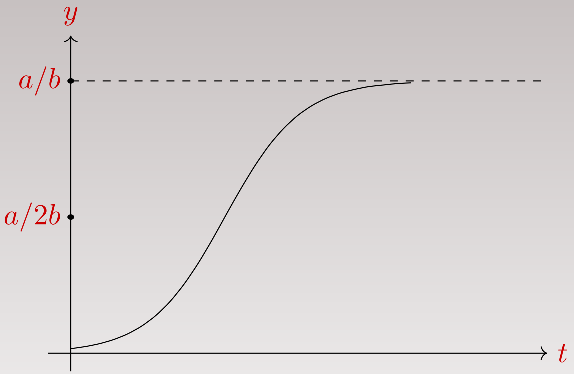


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• The above curve is called a *logistic* or *S-shaped curve*. Note $\frac{a}{b}$ is the limiting population.

## Atomic Waste Disposal Problem, Linear Model

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- Here $W = mg$, the force due to gravity, $B$ is the Buoyancy force of water acting against the forward movement and $D = cV$ is the drag force of water (it is a kind of resistance), where $V = \frac{dy}{dt}$, the velocity of the object

Equivalently

$$\frac{dV}{dt} + \frac{cg}{W}V = \frac{g}{W}(W - B), \; V(0) = 0. \tag{8}$$

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- The limiting value 700 is far above the permitted critical velocity 40. But it is possible that this may be happening when $t$ is very large. Thus it remains to show that $V(t)$ does not reach 40 by the time it reaches the floor. But it is not possible to compute $t$ at which time the drums hit the ocean floor and one needs to do further analysis.

## Atomic Waste Disposal Problem, non-linear model

- The idea is to view the velocity $V(t)$ not as a function of time, but as a function of position $y$. Let $v(y)$ be the velocity at the height $y$ measured from the surface of the ocean. Then $V(t) = v(y(t))$.

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- Solve the above equation to get

$$\frac{gy}{W} = -\frac{v}{c} - \frac{W - B}{c^2} \log \frac{W - B - cv}{W - B}. \tag{11}$$

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- One may not get $V$ explicitly in terms of $y$ as it is a non-linear equation, but can get good estimates of the velocity $v(y)$ at height $y$ and it is estimated that $v(300) \approx 45$ and hence the drum could break at height 300.

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- **Tail to the Tale:** This problem was initiated when the environmentalists and scientists questioned the practice of dumping waste materials by the atomic energy commission of USA. After the study, the dumping of low level atomic waste at sea was forbidden.

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- At equilibrium, the spring has been stretched a distance $\Delta l$ and so $k\Delta l = mg$. Applying Newton's second law, we get

$$m\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt} + F(t). \tag{12}$$

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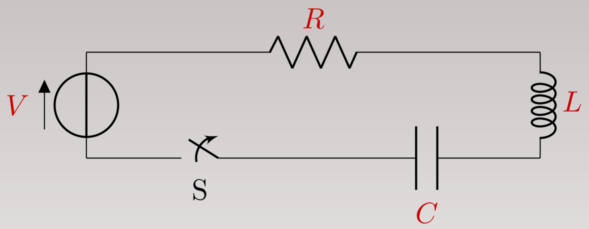
$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t),\ \ m, c, k \geq 0. \tag{13}$$

# RLC Circuit problem

- A basic RLC circuit is given below

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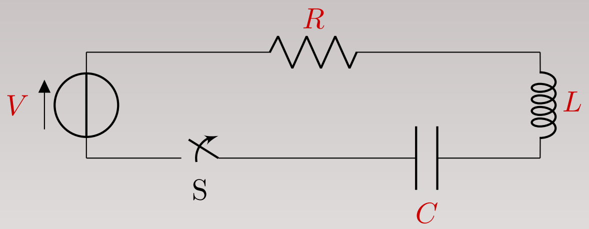


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- The charge $Q$ ( or the current $I$, $E$ replaced by the derivative)

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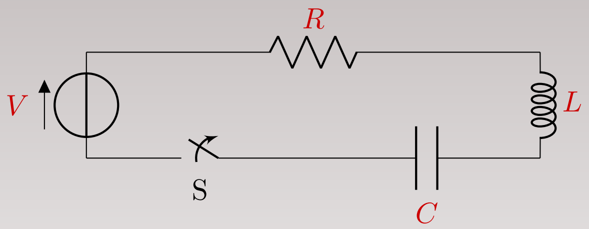


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- Both are second order linear equations with constant coefficients

## Analysis on S-M-D system

- Case (i) **(Free - Undamped vibrations, $F = 0, c = 0$):** In this case the general solution is oscillatory which is given by

$$y(t) = a\cos w_0 t + b\sin w_0 t = R\cos(w_0 t - \delta).$$

Here $w_0 = \sqrt{\frac{k}{m}}$ is the *natural frequency* of the system and $R = \sqrt{a^2 + b^2}$, $\delta = \tan^{-1}(\frac{b}{a})$, $T_0 = \frac{2\pi}{w_0}$ are , respectively, the *amplitude*, *phase angle* and the period of the motion between $-R$ and $R$.

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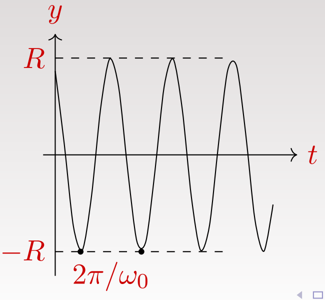

## Analysis on S-M-D system; Conti...

• Case (ii) (**Damped-Free motion**, $F = 0, c \neq 0$). If $r_1, r_2$ are the roots of the characteristic equation, we can write the solution as

$$y(t) = \begin{cases} ae^{r_1 t} + be^{r_2 t} & \text{if } c^2 - 4mk > 0 \\ (a + bt)e^{-ct/2m} & \text{if } c^2 - 4mk = 0 \\ e^{\frac{-ct}{2m}}\left[a\cos\mu t + b\sin\mu t\right] & \text{if } c^2 - 4mk < 0, \end{cases} \tag{15}$$

where $\mu = \frac{\sqrt{4mk - c^2}}{2m}$.

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• Note that in the first two cases $y(t) \to 0$ as $t \to \infty$, and that $y(t)$ creeps back to the equilibrium position and no oscillations as well. These are referred to as *over-damped systems*. The interesting third case is known as *under damped motion* which occurs quite often in mechanical vibrations. Thus, it is a case of damped vibrations.

## Analysis on S-M-D system; Conti...

- In the third case, rewrite the solution as $y(t) = Re^{-ct/2m}\cos(\mu t - \delta)$ which goes to $0$ as $t \to \infty$ and $y(t)$ oscillates between the curves $y = \pm Re^{-ct/2m}$.

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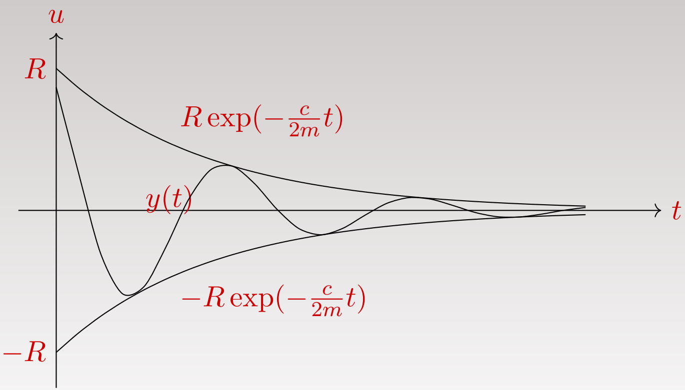


Figure: Free-damped vibration

## Analysis on S-M-D system; Conti...

• As remarked earlier, the motion dies out if there is damping in the system. In other words, the initial disturbance is dissipated by damping. This is why it is very useful in mechanical systems. It can kill undesired vibrations like shocks transmitted in an automobile, principle behind shock absorbers, momentum from gun barrel etc.

## Analysis on S-M-D system; Conti...

- (**Forced - Damped Vibrations**, $F \neq 0, \ c \neq 0$).
- If we take the applied force as $F(t) = F_0 \cos wt, \ F_0 \neq 0, \ c \neq 0$, a particular solution can be obtained as

$$y_p(t) = \frac{F_0 \cos(wt - \delta)}{[(k - mw^2)^2 + (c^2 w^2)]^{1/2}}, \ \delta = \frac{c}{k - mw^2} \tag{16}$$

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- The general solution can be written as

$$y(t) = \varphi + y_p(t), \tag{17}$$

where $\varphi$ is the solution to the homogeneous equation and $\varphi(t) \to 0$ as $t \to \infty$. Thus, for large time $y(t) \sim y_p(t)$. The solution $y_p$ is called **steady state** part of $y(t)$ and $\varphi(t)$ is called the **transient part**.

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- Case (iv): (**Forced-Undamped vibrations**, $c = 0, \ F \neq 0$).

Again take $F = F_0 \cos wt$. In this case $w_0 = \sqrt{\frac{k}{m}}$ is the natural frequency of the system.

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- There are two cases: **Case (iv a) (without resonance),** $c = 0, \ F \neq 0, w \neq w_0$**:** If $w \neq w_0$, that the applied frequency $w$ is different from the natural frequency $w_0$, then the case is similar to the earlier cases and the solution is the sum of two oscillatory functions in the form

$$y(t) = c_1 \cos w_0 t + c_2 \sin w_0 t + \frac{F_0}{m(w_0^2 - w^2)} \cos wt.$$

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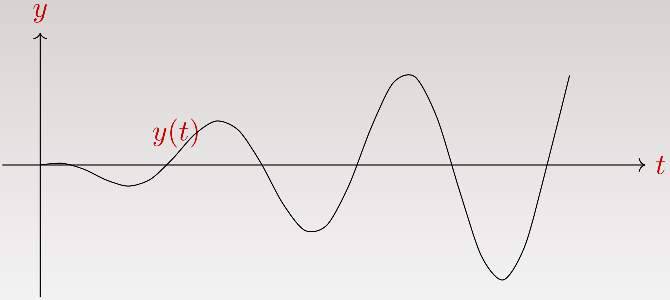


Figure: Forced undamped vibrations with resonance

## Analysis on S-M-D system; Conti...

• This is the reason for the collapse of the Tacoma bridge on November 7, 1940 at 11.00 am. The system needs to have sufficient damping. This is also responsible for the collapse of the Broughton suspension bridge near Manchester. This occurred when a column of soldiers marched in cadence over the bridge, there by setting up a periodic force rather large amplitude. The frequency was equal to the natural frequency. Thus large oscillations were induced and the bridge collapsed. It is for this reason that soldiers are ordered to break cadence when crossing a bridge.

## Analysis on S-M-D system; Conti...

- Among many similarities with mechanical vibrations, electrical circuits also have the property of resonance. Unlike mechanical systems, resonance is put to good use here like, the tuning knob of radio is used to vary the capacitance in such a manner that the resonant frequency is changed until it agrees with the frequency of external signal. The amplitude of the current produced by this signal will be much greater than that of other signals so that we get the desired signal.

# Duffing equation

• Duffing equation, named after George Duffing is a non-linear second order equation used to model certain damped and driven oscillators. If $x(t)$ is the displacement, it satisfies equatio

$$\ddot{x} - \alpha x + \beta x^3 + \delta \dot{x} = \gamma \cos(\omega t).$$

Here $\alpha$ is stiffness, $\beta$ is the amount of non-linearity in the restoring force, $\delta$ damping, $\gamma$, the amplitude and $\omega$ the frequency. RHS is the driving force.

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• If $\beta = \delta = 0$, it is nothing but the simple harmonic motion. In general, it models more complicated potentials like; spring pendulum whose spring stiffness does not obey exactly Hook's Law.

## Van der Pol Equation or Oscillator

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- This equation, apparently first introduced in 1896 by Lord Rayleigh, was extensively studied both theoretically and experimentally using electric circuits by van der Pol when he was working for the Philips company in Sweden around 1927. He also studied this equation with forced periodic term like $A \sin \omega t$ and observed what we call today *chaos*.
- A detailed mathematical analysis of this equation was done by Cartwright and Littlewood[1945] and by Levinson[1949]; their study revealed the existence of the paradoxical combination of randomness and structure, which is also called *deterministic chaos* in the current literature.

## Lorenz System

- The Lorenz system is given by ($R, \sigma, b$ are fixed parameters)

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= Rx - y - xz \qquad (18) \\ \dot{z} &= -bz + xy \qquad (19) \end{aligned}$$

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- Motivated by the meteorological problem of weather prediction, Lorenz derived these equations as a much simplified model of Rayleigh-Bernard convection in fluids, which provided a first specific example of chaotic dynamics persisting for all time. Later, in Japan, Ueda studied the steady state chaotic behaviour of forced Duffing oscillator.

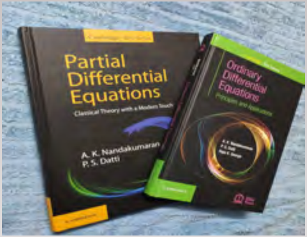
## Lotka-Volterra Prey-Prediction Model

• The dynamical behaviour governing the growth, decay and general evolution of interacting biological species is modelled by the Lotka-Volterra prey-predator equations given as

$$\dot{x} = ax - bxy \tag{20}$$

$$\dot{y} = cxy - dy \tag{21}$$

Here $x$ denotes the population of the prey, say rabbits at a given time and $y$ denotes the population of the predator,say foxes. The constants $a, b, c, d$ are all positive and represent the growth and decay rates of prey and predator.


Partial
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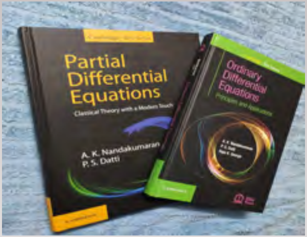
# *Thank You!* A. K. Nandakumaran, IISc., Bangalore

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