Role of Analysis and Linear Algebra in Differential Equations 2

Stability of Linear Systems

A. K. Nandakumaran

Department of Mathematics
Indian Institute of Science, Bangalore 560012.
Email: nands@iisc.ac.in

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- A general n^{th} order linear system of ODEs can be written in the form

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t) \tag{1}$$

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- For example initial conditions can be of the form

$$y(0) = y_0,\ y^{(1)}(0) = y_1, \ldots y^{(n-1)}(0) = y_{n-1}.$$

If $g(t) = 0$, the equation (1) is homogeneous linear, otherwise it is called non-homogeneous.

n^{th} Order Linear System, Conti...

- By introducing new variables:

$$x_1 = y, x_2 = y^{(1)}, \cdots, x_n = y^{(n-1)}(t) = \frac{d^{(n-1)}y}{dt^{(n-1)}},$$

one can convert the equations (1) into a first order system of n equations which has a matrix representation $\dot{x}(t) = Ax(t) + G(t)$ where

$$A = A(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -p_n & -p_{n-1} & \dots & \dots & -p_1 \end{bmatrix}, x(t) = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_{n-1} \\ x_n \end{bmatrix}, G(t) = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ g(t) \end{bmatrix}$$

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- A general system will be

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- If A depends on t, (2) is called a *non-autonomous system* and if $A(t) = A$ is independent of t, it is called an *autonomous system*.

- The existence and uniqueness can be obtained directly from the general theory under suitable continuity assumptions on $A(t)$ and $G(t)$. It can also be derived directly.

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- The main part of this talk is to study the stability of 2×2 systems via the diagonalization of matrices.

- Linear algebra, eigenvalues, eigenvectors, diagonalization etc. play a fundamental role. It also motivates these notions in linear algebra.

Definition of e^{tA}

- Given a matrix A, it can be viewed as a *linear operator* $A : \mathbb{R}^n \to \mathbb{R}^n$ whose norm is given by $||A|| = \sup_{|x|=1} |Ax|$, where $|\cdot|$ is the norm (or modulus) in $\mathbb{R}^n$.

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• One can consider the partial sums of operators $A_m = \sum_{k=0}^{m} t^k \frac{A^k}{k!}$ and $||A_m|| \leq e^{||A||T}$, $0 \leq t \leq T$ and prove that A_m converges to a linear operator from $\mathbb{R}^n \to \mathbb{R}^n$ whose matrix representation is denoted by e^{tA}.

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- In other words, $e^{tA} = \displaystyle\sum_{k=0}^{\infty} t^k \frac{A^k}{k!}$.

- **Exercise** Compute $\frac{d}{dt}(e^{(t-t_0)A}x_0)$, justify and show that $x(t) = e^{(t-t_0)A}x_0$ is a solution to the system $\dot{x}(t) = Ax(t)$, $x(t_0) = x_0$. Further, directly show that it is the unique solution.

Computation of e^{tA} in Special Cases

- Even the solution has explicit representation as $x(t) = e^{tA}x_0$, the computation of e^{tA} is not easy and in fact, it does not reveal anything about the behaviour of trajectories. This is more interesting in application than a mere representation.

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- If $A = diag(\lambda_1, \lambda_2, ...\lambda_n)$ is a diagonal matrix, then

$$e^A = diag(e^{\lambda_1}, e^{\lambda_2}, ...e^{\lambda_n})$$

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- (Similarity) Suppose $C = PAP^{-1}$ for some invertible P, Then $e^C = Pe^AP^{-1}$. In particular, if C is diagonalizable, that is $C = diag(\lambda_1, \lambda_2, ...\lambda_n)$, then

$$e^A = e^{P^{-1}CP} = P^{-1}e^CP = P^{-1}diag(e^{\lambda_1}, e^{\lambda_2}, ...e^{\lambda_n})P$$

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- If A not diagonalizable, then what? *Jordan decomposition*

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$$(C1)\;\; B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \;\; (C2)\; C = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \;\; (C3)\; D = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

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- $(C3)$ if A has two complex eigenvalues $a + ib$ and $a - ib$.

Exponential of the Special matrices

- One can compute the exponential of the above matrices easily as

$$e^B = \begin{bmatrix} e^\lambda & 0 \\ 0 & e^\mu \end{bmatrix}, \quad e^C = e^\lambda \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad e^D = e^a \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b). \end{bmatrix}$$

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- This will allow us to write e^{tB}, e^{tC} and e^{tD} and hence the solutions to the differential systems corresponding to the above class of matrices $e^{tB}x_0$, $e^{tC}x_0$ and $e^{tD}x_0$

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- We have

$$e^{tB} = \begin{bmatrix} e^{t\lambda} & 0 \\ 0 & e^{t\mu} \end{bmatrix}, \quad e^{tC} = e^{t\lambda} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad e^{tD} = e^{ta} \begin{bmatrix} \cos(tb) & -\sin(tb) \\ \sin(tb) & \cos(tb). \end{bmatrix}$$

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• From a dynamic system point of view $x(t)$ describes the motion of a particle in x_1x_2-plane which we refer as *phase plane* and t is treated as time. At $t = 0$, the particle is at x_0 and particle moves to $x(t)$ at time t.

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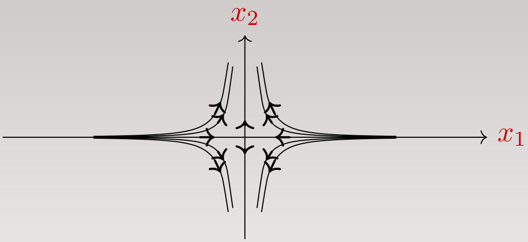
• The representation of all solution curves in the phase plane (called *phase space* in higher dimensions) $\mathbb{R}^n$ is known as *phase portrait*.

Saddle point equilibrium

- The following figure gives the various trajectories of the particle under motion, where the arrow represents the direction of motion.



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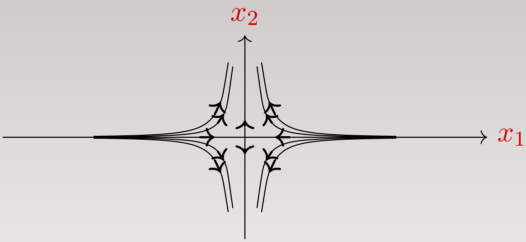

Figure: Saddle Point Equilibrium

• This is the feature of any system of the form $(C1)$ with different signs for λ and μ except that the direction of arrows might change.

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- Thus a *dynamical system* is a mapping $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ given by the solution $x(t, x_0)$, that is $\Phi(t, x_0) = x(t, x_0)$.

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- Another notion is a vector field (x, Ax)

Equilibrium Points

- For a general system $\dot{x} = f(t, x)$, a point x_0 is called an *equilibrium point* if $f(t, x_0) = 0$. It is an equilibrium in sense that $x(t) \equiv x_0$ is a solution to the ODE system which starts at x_0.

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• Now we move on to study 2×2 systems and it is easy to have complete analysis as there are only two eigenvalues.

Equilibrium: Node; Case $(C1)$

- We consider the case of $(C1)$, but same sign for λ and μ. For, take $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then the solution is given by $x_1(t) = x_{01}e^{2t},\ x_2(t) = x_{02}e^{t}$. Eliminating t, we will get $x_1 = cx_2^2$.

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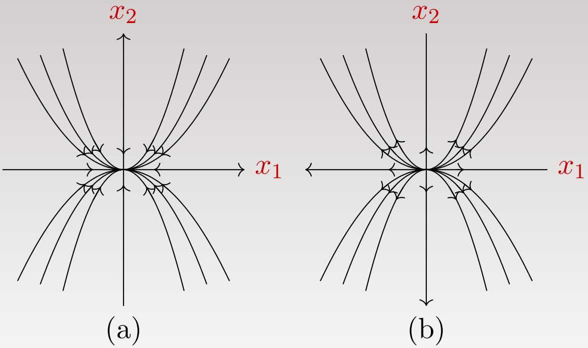

Figure: (a) Stable Node, (b) Unstable Node

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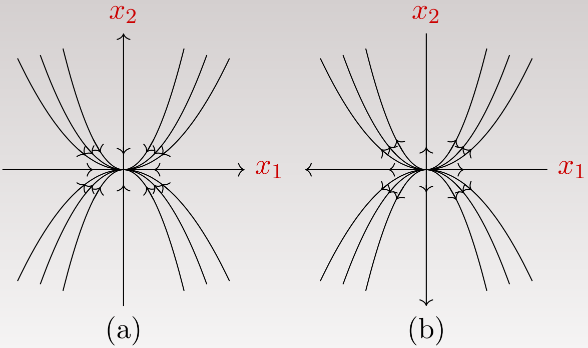

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Equilibrium: Node, Case $(C2)$

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$$x_1(t) = (x_{01} + x_{02}t)e^{t\lambda},\ x_2(t) = x_{02}e^{t\lambda}$$

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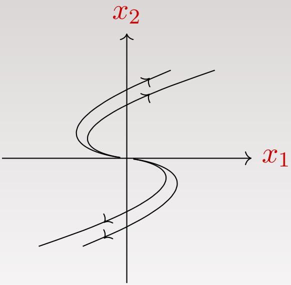

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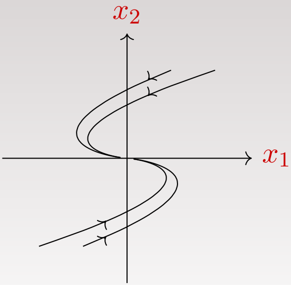

(b) Stable Node

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- Thus $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and the solution is

$$x(t) = e^{at} \begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix} x_0$$

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- Indeed $sign(a)$ will determine the stability, whereas the matrix components causes the periodicity with $sign(b)$ determining the orientation. The phase portrait with $a \neq 0$ and $a = 0$ are given below.

Equilibrium: Unstable Focus, Case $(C3)$ with $a > 0$

- The solution goes to infinity from any initial value rotating around the origin.

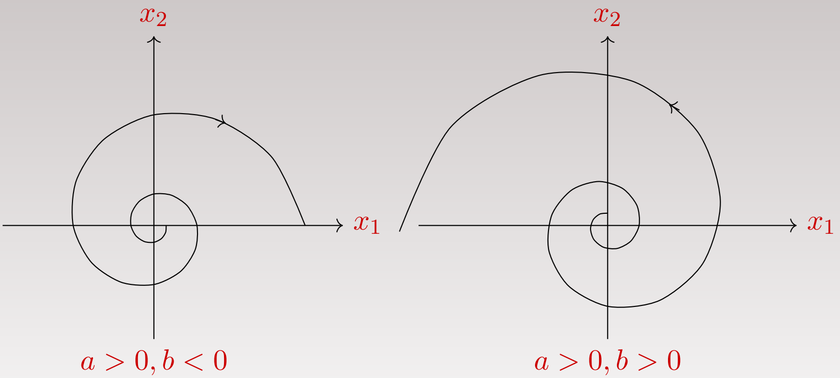

Figure: Unstable Focus

Equilibrium: Stable Focus, Case (C3) with $a < 0$

- The solution goes to 0 from any initial value rotating around the origin.

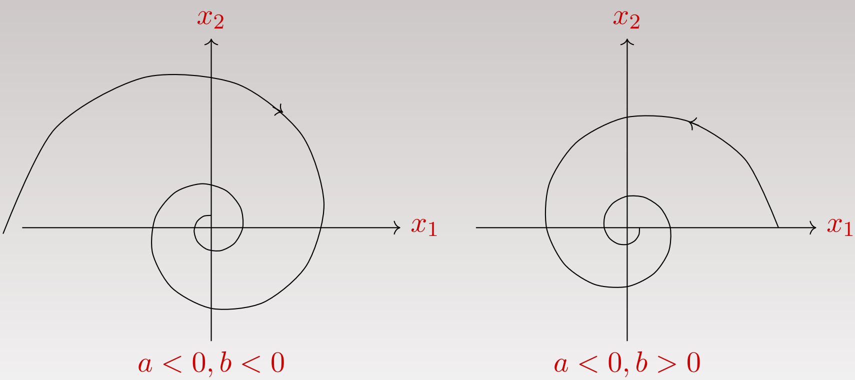

Figure: Stable Focus

Equilibrium: Center, Case $(C3)$ with $a = 0$

- Pure periodic rotations in the case of pure imaginary eigenvalues.

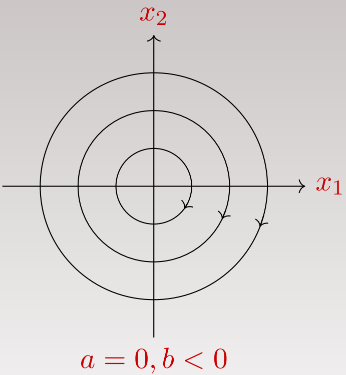

Figure: Centre

Degenerate Case; with 0 Eigenvalue

• So far we have analyzed the situation when there are no 0 eigenvalues. When the matrix A has 0 eigenvalue, then the equation $Ax = 0$ will have more than one non-trivial solution which will be one or two dimensional subspace.

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- Each of these non-trivial solution will be an equilibrium point

- Consider the case $A = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$, then $x_1(t) = x_{01}, \; x_2(t) = x_{02}e^{-2t}$.

In this degenerate case, all the points on the x_1-axis are equilibrium points. (A point x_0 is called an *equilibrium point* of the dynamical system $\dot{x} = f(x)$ if $f(x_0) = 0$.)

Degenerate Case; with 0 Eigenvalue

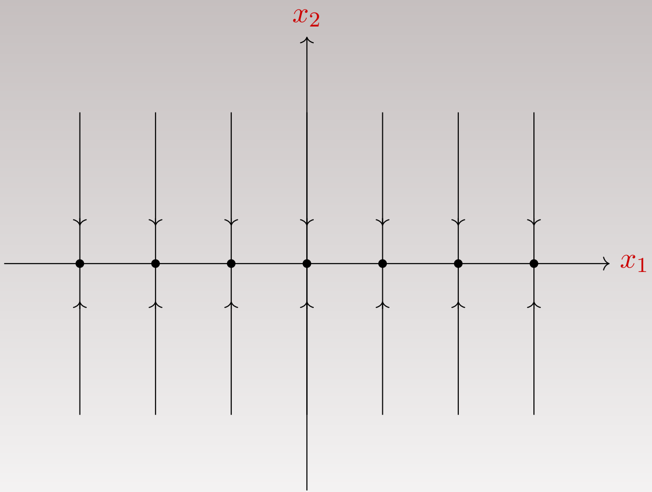

Figure: Degenerate Case

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- Two matrices A and B are said to be *linearly equivalent* if there is an invertible matrix P such that $B = P^{-1}AP$ or $A = PBP^{-1}$.

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$$\dot{x} = Ax \;\; \Rightarrow \;\; \dot{x} = PBP^{-1}x, \; x(0) = x_0$$

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- Thus the solution x is given by

$$x(t) = Py(t) = Pe^{Bt}y_0 = Pe^{Bt}P^{-1}x_0.$$

General Theorems, Conti...

- For a 2×2 system, the general *Jordan form* gives the following theorem.

Theorem

Given a 2×2 matrix A, there is an invertible matrix P such that $A = PBP^{-1}$ and B takes one of the forms of $(C1)$, $(C2)$ or $(C3)$.

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Remark

The stability and nature of the solution trajectories do not change under linear equivalence.

Linear Equivalence

Definition

A linear system (3) is said to have a saddle, a node, a focus or a center at the origin, respectively, if the matrix linearly equivalent to

$$(I)\ \ B_1 = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix},\ \lambda\mu < 0$$

$$(II) B_1, \lambda\mu > 0\ \ or\ \ B_2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},\ \lambda \neq 0$$

$$(III)\ B_3 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, with\ a \neq 0,\ b \neq 0$$

$$(IV)\ B_4 = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix},\ b \neq 0$$

Bifurcation Diagram

- A schematic representation (Bifurcation diagram) is shown below connecting the determinant and trace of A. Let $\delta = det(A)$, $\tau = trace(A)$ and $\Delta = \tau^2 - 4\delta$

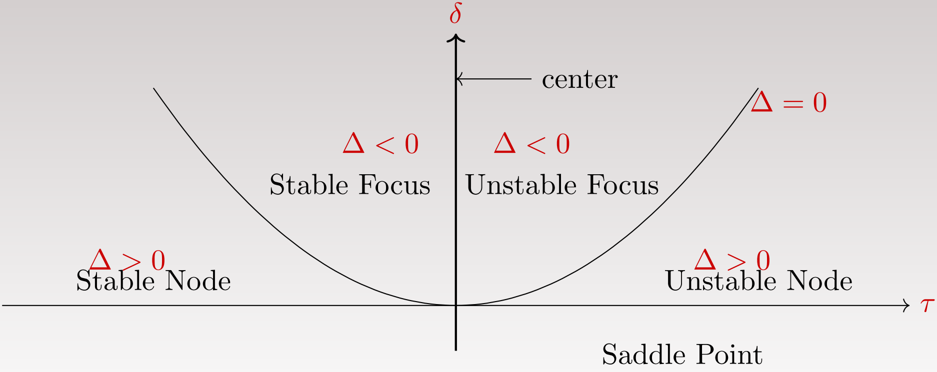

Figure: Bifurcation Diagram

An Example

- Consider the system

$$\dot{x}_1 = -x_1 - 3x_2, \quad \dot{x}_2 = 2x_2$$

The matrix $A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1, \ \lambda_2 = 2$ with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

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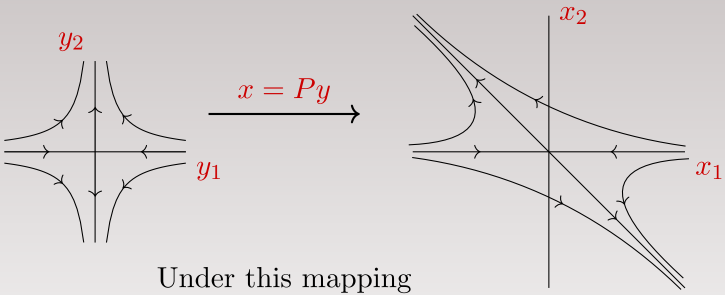
- Hence $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

- Further $B = P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \dot{y_1} = -y_1, \ \dot{y_2} = 2y_2,$ which gives a diagonal system equivalent to the given system. Note that the stability do not change. Moreover, since it is a linear correspondence, the x and y axes will corresponds to lines passing through the origin.

An Example, Conti...

- The phase portrait can be depicted as follows.



Under this mapping

$$P\begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} y_1 \\ 0 \end{bmatrix}, \quad P\begin{bmatrix} 0 \\ y_2 \end{bmatrix} = \begin{bmatrix} -y_2 \\ y_1 \end{bmatrix}$$

Higher Dimensional Systems

• When the eigenvalues of A are distinct, then analysis and the representation of the solution is quite straight forward.

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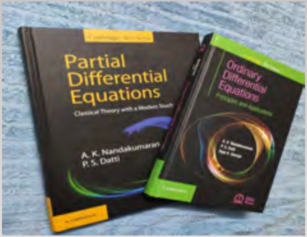
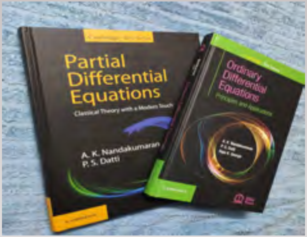
• In general, one can appeal to the Jordan decomposition which decomposition using the eigenvalues. One need to classify eigenvalues with negative real part, 0 real part and positive real part to obtain subspaces $\mathbb{R}^n$ which are stable, center and unstable . The stable and unstable subspaces are invariant under the flow.

Thank You! A. K. Nandakumaran, IISc., Bangalore





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