Power Series Solutions And Special Functions: Review of Power Series

Pradeep Boggarapu

Department of Mathematics

BITS PILANI K K Birla Goa Campus, Goa

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Finding the general solution of a linear differential equation

$$y'' + P(x)y' + Q(x)y = R(x)$$

depends on determining any two linear independent solutions of the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0$$

So far, we have a systematic procedure for constructing fundamental solutions (linear independent solutions of associated homogeneous equation), if the equation has constant coefficients (i.e., P(x) and Q(x) are constant functions).

For the equations with variable coefficients we don't have any method to find fundamental solutions expect the case of knowing one solution where we use the known solution to find another solution via the formula $y_2(x) = v(x)y_1(x)$ with

$$v(x) = \int \frac{1}{y_1^2} e^{-\int P(x)dx} dx$$

Examples: Let p, a, b, c and k are real constants,

1. Bessel's Differential Equation: $x^2y'' + xy' + (x^2 - p^2)y = 0$
2. Legender's Differential Equation: $(1 - x^2)y'' - 2xy' + p(p - 1)y = 0$
3. Hermite Differential Equation: $y'' - 2xy' + 2py = 0$
4. Guass Hypergeometric Differentila Equation:

$$x(x - 1)y'' + [c - (a + b + 1)x]y' - aby = 0$$

5. Laguerre's Differential Equations: $xy'' + (1 - x)y' + py = 0$
6. Airy's Equation: $y'' \pm p^2xy = 0$

In most of the cases the solutions of the above differential equations are beyond the elementary functions which are called as *special functions*.

Many of the special functions find applications in connection with the partial differential equations of mathematical physics. Thare are also important in modern pure mathematics, through the theory of orthogonal expansions.

For a larger class of linear diffrerential equations with variable coefficients such as above equations, we must search for solutions beyond the familiar elementary functions of calculus.

The principal tool we need is the representation of a given function by a power series.

Then, we assume that the solutions y have power series representations $\sum_{n=0}^{\infty} a_n(x - x_0)^n$, and then determine the coefficients a_n's so as to satisfy the differential equation similar to the method of undetermined coefficients.

Power series about the point zero: It is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{0.1}$$

where $a_0, a_1, a_2, \ldots, a_n, \ldots$ are real constants.

Power series about the point x_0: It is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + \cdots \tag{0.2}$$

where $a_0, a_1, a_2, \ldots, a_n, \ldots$ are real constants.

Examples: $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots ;$

$$\sum_{n=0}^{\infty} x^n = 1 + x^1 + x^2 + x^3 + \cdots .$$

Convergence of power series: The power series is said to converge at a point x if its n-th partial sum $\sum_{k=0}^{n} a_k x^k$ converges that is to say that the limit $L = \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k$ exists.

In this case the *sum* of the series is the limit i.e. $L = \sum_{n=0}^{\infty} a_n x^n$ and such points x are called points of convergence.

Note that $x = 0$ is always a point of convergence of the power series (0.1). See the following examples

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots ; \tag{0.3}$$

$$\sum_{n=0}^{\infty} x^n = 1 + x^1 + x^2 + x^3 + \cdots ; \tag{0.4}$$

$$\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \cdots . \tag{0.5}$$

The first series converges for every value of x in $\mathbb{R}$; second converges only for $|x| < 1$ and the third series diverges for all $x \neq 0$.

The power series in x that behaves like third series are of no interest to us.

Fact: The points of convergence of a power series (0.1) (or (0.2)) form an interval. Moreover there exists $0 \leq R \leq \infty$ such that the power series (0.1) (or (0.2)) converges for all $|x| < R$ (resp. $|x - x_0| < R$) and diverges for all $|x| > R$ (resp. $|x - x_0| > R$). Here R is called *radius of convergence*.

In many cases the radius of convergence can be found by using the following formulas,

$$R = \lim_{n \to 0} \left| \frac{a_n}{a_{n+1}} \right| \qquad \text{or} \qquad R = \lim_{n \to 0} \frac{1}{\sqrt[n]{|a_n|}}$$

whenever the limits exist.

Regardless of the existence of the above limits, it is known that R always exists.

Differentiation of power series: Suppose that the power series (0.1) converges for $|x| < R$ with $R > 0$ and denote the sum by $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Then $f(x)$ is automatically is continuous and has the derivatives of all orders for $|x| < R$. Also,

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots , \tag{0.6}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) x^{n-2} = 2a_2 + 3 \cdot 2a_3 x + \cdots , \tag{0.7}$$

and so on, and each of the resulting series converges for $|x| < R$.

And we can link the coefficient a_n to $f(x)$ and its derivative via the following formula

$$a_n = \frac{f^{(n)}(0)}{n!} \tag{0.8}$$

Algebra of power series: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be two power series with radius of convergence at least $R > 0$, then these power series can be added or subtracted termwise:

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n = (a_0 \pm b_0) + (a_1 \pm b_1)x + \cdots .$$

They can also be multiplies as they were polynomials, in the sense that

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$.

If $f(x) = g(x)$ for $|x| < R$ if and only if $a_n = b_n$ for all n i.e. If both series converges to the same function for $|x| < R$ if and only if they have the same coefficients.

Power series representation of a function: Let f be a continuous function defined for $|x| < R$ or $(-R, R)$ which has derivatives of all orders in the interval $(-R, R)$. Can $f(x)$ be represented by a power series about the point zero? Equivalently will the following hold

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots \qquad (0.9)$$

throughout the interval $(-R, R)$?

This is often true, but unfortunately it is some times false. The above expansion is valid if the error term $R_n(x)$ in Taylor's formula:

$$f(x) = \sum_{n=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x)$$

convegres to zero as n tends to infity.

By means of the procedure explained in the previous slide, it is quite easy to obtaion the following familiar expansions,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots ; \tag{0.10}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots ; \tag{0.11}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^{2n} \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots ; \tag{0.12}$$

$$\frac{1}{1 \pm x} = \sum_{n=0}^{\infty} (\pm 1)^n x^n = 1 \pm x + x^2 \pm x^3 + \cdots ; \tag{0.13}$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots ; \tag{0.14}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots ; \tag{0.15}$$

The function f for which the above series expansion (0.9) is valid for some neighbourhood of zero is said to be *analytic at $x = 0$.* More generally the analyticity at any point is defined as follows.

Analytic at a point: A function $f(x)$ with the property that a power series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

valid in some neighbourhood of the point x_0 is said to be analytic at x_0.

In this case the coefficients a_n's are necessarily given by

$$a_n = \frac{f^{(n)}(x_0)}{n!},$$

and the above series is called the Taylor series of $f(x)$ at x_0.

Facts about analytic functions:

- Polynomials and the functions e^x, $\sin x$ and $\cos x$ are analytic at all points.
- If $f(x)$ and $g(x)$ are analytic at x_0, then $f(x) + g(x)$, $f(x)g(x)$, and $f(x)/g(x)$ [if $g(x_0) \neq 0$] are also analytic at x_0 .
- If $f(x)$ is analytic at x_0, $f'(x_0) \neq 0$ and $f^{-1}(x)$ is a continuous inverse, then $f^{-1}(x)$ is analytic at $f(x_0)$.
- If $g(x)$ is analytic at x_0 and $f(x)$ is analytic at $g(x_0)$, then $f \circ g(x) = f(g(x))$ is analytic at x_0.
- The sum of a power series is analytic at all points inside the interval of convergence.

Series solutions of first order equations: We have repeatedly emphasized that many interesting and important differential equations cannot be solved by any of the methods discussed so far.

And also note that solutions for equations of this kind can often be found in terms of power series.

Our purpose in this section is to discuss that how we use power series representation to solve a differential equation by demonstrating an example with a first order equations that are easy to solve by elementary methods.

Problem 1: Solve the first order differential equation $y' = y$ by using power series representation.

Solution: We assume that the given equation has a power series solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{0.16}$$

that converges for $|x| < R$ with $R > 0$.

That is, we are assuming that the differential equation has a solution say y that is analytic at the origin. We know that

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots , \quad (0.17)$$

in the interval of convergence. Since $y' = y$, we have that

$$\begin{aligned} \sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty} a_n x^n \\ a_1 + 2a_2 x + 3a_3 x^2 + \cdots &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots . \end{aligned}$$

The above both series must have the same coefficients:

$$a_1 = a_0, \quad 2a_2 = a_1, \quad 3a_3 = a_2, \cdots (n+1)a_{n+1} = a_n, \cdots .$$

These equations enables us to express each a_n in terms of a_0:

$$a_1 = a_0, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{2 \cdot 3}, \cdots , a_n = \frac{a_0}{n!}, \cdots .$$

If you substitute all these values of coefficients in (0.16), we obtain our power series solution

$$y = a_0\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right)$$

We can easily recognise that the above series as the power series expansion of e^x, so the above can be written as

$$y = a_0 e^x.$$

, This example suggests a useful method for obtaining the power series expansion of a given function:

Find the differential equation with initial condition satisfied by the function, and then solve this equation by power series. We consider an example (to understand this idea).

Proplem 2: Find the power series expansion of $y = (1+x)^p$ about the origin (where p is a constant) by using the method described in the above. Use the result to show that

$$\sqrt{2} = 1 + \frac{1}{2}\cdot\frac{1}{2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{2^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{2^3} + \cdots.$$

Part 1: First find a differential equation satisfied by y:

$$(1+x)y' = py, \qquad y(0) = 1$$

As before we assume that the above equation has a power series solution

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{0.18}$$

with positive radius $R > 0$ of convergence. It follows that

$$\begin{aligned} y' &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots, \\ xy' &= a_1 x + 2a_2 x^2 + \cdots + na_n x^n + \cdots, \\ py &= pa_0 + pa_1 x + pa_2 x^2 + \cdots + pa_n x^n + \cdots. \end{aligned}$$

Since the d.e. is $(1+x)y' = py$, the sum of the first two series must equal the third, so equating the coeffiecients of the successive powers of x gives

$$a_1 = pa_0, \qquad 2a_2 + a_1 = pa_1, \qquad 3a_3 + 2a_2 = pa_2, \ldots,$$
$$(n+1)a_{n+1} + na_n = pa_n, \ldots$$

The initial condition $y(0) = 1$ implies that $a_0 = 1$, so

$$a_1 = p, \quad a_2 = \frac{a_1(p-1)}{2} = \frac{p(p-1)}{2},$$

$$a_3 = \frac{a_2(p-2)}{3} = \frac{p(p-1)(p-2)}{2 \cdot 3}, \dots,$$

$$a_n = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}, \dots.$$

With these coefficints, the solution (0.18) becomes

$$\begin{aligned} y =& 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \\ &+ \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}x^n + \cdots. \end{aligned} \tag{0.19}$$

To conclude that (0.19) actually is the desired solution, it suffices to observe that the series converges for $|x| < R$ for some $R > 0$.

We can show that $R = 1$ by using the ratio tests or by using first of the formulas given previously

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to 0} \left| \frac{n+1}{p-n} \right| = 1.$$

On comparing the two solutions $y = (1+x)^p$ and (0.19), and using the fact that the initial value problem has only one solution, we have

$$\begin{aligned}(1+x)^p =& 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \\ &+ \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}x^n + \cdots \end{aligned} \tag{0.20}$$

for $|x| < 1$. This expansion is called the binomial series and this formula generalises the binomial theorem to the case of an arbitrary exponent.

Part2: Now we substitute $x = -\frac{1}{2}$ and $p = -\frac{1}{2}$ in the power series (0.20) to get the required indentity.

$$\left(1-\frac{1}{2}\right)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)\cdot\left(-\frac{1}{2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\cdot\left(-\frac{1}{2}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\cdot\left(-\frac{1}{2}\right)^3 + \cdots.$$

After simplying the above you get the required identity

$$\sqrt{2} = 1 + \frac{1}{2}\cdot\frac{1}{2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{2^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{2^3} + \cdots.$$

Problem 3: Express $\sin^{-1} x$ in the form of a power series $\sum a_n x^n$ by solving $y' = (1-x^2)^{-1/2}$ in two ways. (*Hint*: Remember the binomial series.) Use this result to obtain the formula

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3\cdot 2^3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5\cdot 2^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7\cdot 2^7} + \cdots .$$

Solution:

Part 1: We first find an intial value problem satisfied by $y = \sin^{-1} x$:

$$y' = (1-x^2)^{-1/2}, \qquad y(0) = 0$$

As before we assume that the above equation has a power series solution

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{0.21}$$

with positive radius $R > 0$ of convergence. It follows that

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots + (n+1)a_{n+1}x^n + \cdots \tag{0.22}$$

We know that

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots + \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}x^n + \cdots .$$

Now we replace p by $-1/2$ and x by $-x^2$ in the above to get

$$\left(1-x^2\right)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}\cdot x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot x^6 + \cdots + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot x^{2n} + \cdots .$$

Since the d.e. is $y' = (1-x^2)^{-1/2}$, so equating the coeffiecients of the successive powers of x on both series gives

$$2a_2 = 4a_4 = a_{2n} = \cdots = 0; \quad a_1 = 1; \quad 3a_3 = \frac{1}{2}; \quad 5a_5 = \frac{1\cdot 3}{2\cdot 4};$$

$$7a_7 = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}; \quad \ldots; \quad (2n+1)a_{2n+1} =, \ldots \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$$

Further simplification implies

$$2a_2 = 4a_4 = a_{2n} = \cdots = 0; \quad a_1 = 1; \quad a_3 = \frac{1}{2}\cdot\frac{1}{3}; \;\; a_5 = \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5};$$
$$a_7 = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7}; \quad \ldots; \quad a_{2n+1} =, \ldots \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{1}{2n+1}$$

Since $y(0) = 0$, $a_0 = 0$, now substitute all the values of coefficients in power series solution (0.21) and the initial value problem has only one solution which means the power series solution is nothing but $\sin^{-1} x$, therefore

$$\sin^{-1} x = x + \frac{1}{2}\cdot\frac{1}{3}\,x^3 + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}\,x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7}\,x^7 + \cdots .$$

If $x = \frac{1}{2}$ then $\sin^{-1} x = \frac{\pi}{6}$ and hence in view of the above we get the required identity

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3\cdot 2^3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5\cdot 2^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7\cdot 2^7} + \cdots .$$

Problem 4: Find the power series solution of the initial value problem

$$(1+x)y' = 1; \qquad y(0) = 0,$$

and also find solution of the same by using variable separable method to get the follwoing identity

$$\log_e 2 = \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2^2} + \frac{1}{3} \cdot \frac{1}{2^3} + \frac{1}{4} \cdot \frac{1}{2^4} + \cdots .$$

Problem 5: Find the power series solutions of the each of the following first order differentail equations:

(1) $y' - y = 0$ (2) $y' = e^{x^2} y$ (3) $y' - xy = 0$

(4) $(1-x)y' = y$ (5) $y' - y = x^2$ (6) $y + xy = 1 + x$

(7) $(1+x^2)y' = 0$.

Second order linear equations. Ordinary points:

We have seen the power series solutions for first order linear equations. We now turn our attention to the general homogeneous second order linear equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{0.23}$$

As we know, it is occasionally possible to solve such an equation in terms of familiar elementary functions.

For instance, when $P(x)$ and $Q(x)$ are constants and in few other cases as well.

For most of the equations having the greatest significance in pure and applied mathematics can only be solved by power series.

The central fact about the equation (0.23) is that the behaviour of the coefficients $P(x)$ and $Q(x)$ near a point x_0 determines the behaviour its solutions near this point.

In this section we confine ourselves to the case in which $P(x)$ and $Q(x)$ are "well behave" in the sense of being analytic at x_0 that is the case where x_0 is an *ordinary point*.

Ordinary point : If each of $P(x)$ and $Q(x)$ has a power series expansion valid in some neighbourhood of a point x_0, then x_0 is called an *ordinary point* of the equation (0.23). In this case all solutions of equation (0.23) will have power series expansion valid in a neighbourhood of x_0.

In other words, if the coefficient funtions $P(x)$ and $Q(x)$ of equation (0.23) are analytic at a point x_0 then its all solutions are also analytic at this point.

Any point that is not ordinary point of (0.23) is called *singular point*. We will see a bit more about *singular points* after few slides.

Examples: Let p, a, b, c and k are real constants, find out all ordinary and sigular points of the following equations.

1. Bessel's Differential Equation: $x^2y'' + xy' + (x^2 - p^2)y = 0$
2. Legender's Differential Equation: $(1 - x^2)y'' - 2xy' + p(p - 1)y = 0$
3. Hermite Differential Equation: $y'' - 2xy' + 2py = 0$
4. Guass Hypergeometric Differentila Equation:

$$x(x - 1)y'' + [c - (a + b + 1)x]y' - aby = 0$$

5. Laguerre's Differential Equations: $xy'' + (1 - x)y' + py = 0$
6. Airy's Equation: $y'' \pm p^2xy = 0$

We now illustrate the above we said by an example of familiar equations

$$y'' + y = 0 \tag{0.24}$$

The coefficients $P(x) = 0$ and $Q(x) = 1$ are analytic at all the points, we expect the solutions are also analytic at all the points.

So we seek a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{0.25}$$

Differentiating the above, yields

$$\begin{aligned} y'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots, \\ y''(x) &= 2a_2 + 3 \cdot 2a_3 x + \cdots + (n+1)(n+2)a_{n+2}x^n + \cdots, \end{aligned} \tag{0.26}$$

Since $y'' + y = 0$, on adding the above two series term by term and equating to zero, we get

$$\begin{aligned} (2a_2 + a_0) + (2 \cdot 3a_3 + a_1)x + (3 \cdot 4a_4 + a_2)x^2 + (4 \cdot 5a_5 + a_3)x^3 \\ + \cdots + [(n+1)(n+2)a_{n+2} + a_n]x^n + \cdots = 0; \end{aligned}$$

now equating the coefficients of successive powers of x gives

$$2a_2 + a_0 = 0; \quad 2 \cdot 3a_3 + a_1 = 0; \quad 3 \cdot 4a_4 + a_2 = 0;$$
$$4 \cdot 5a_5 + a_3 = 0; \cdots ; (n+1)(n+2)a_{n+2} + a_n = 0; \ldots .$$

By means of these equations we can express a_n in terms of a_0 or a_1 according as n is even or odd:

$$a_2 = -\frac{a_0}{2}; \quad a_3 = -\frac{a_1}{2 \cdot 3}; \quad a_4 = -\frac{a_2}{3 \cdot 4} = \frac{a_0}{2 \cdot 3 \cdot 4};$$
$$a_5 = -\frac{a_3}{4 \cdot 5} = \frac{a_1}{2 \cdot 3 \cdot 4 \cdot 5}; \ldots .$$

With these coefficients, (0.25) becomes

$$\begin{aligned} y &= a_0 + a_1 x - \frac{a_0}{2}x^2 - \frac{a_1}{2 \cdot 3}x^3 + \frac{a_0}{2 \cdot 3 \cdot 4}x^4 + \frac{a_1}{2 \cdot 3 \cdot 4 \cdot 5}x^5 - \cdots \\ &= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \end{aligned} \tag{0.27}$$

where $y(0) = a_0$ and $y'(0) = a_1$.

Let $y_1(x)$ and $y_2(x)$ denote the two series in parentheses. We have shown formally that (0.27) satisfies (0.24) for any two constants a_0 and a_1.

In particular, by choosing $a_0 = 1$ and $a_1 = 0$ we see that y_1 satisfies this equation, and the choice $a_0 = 0$ and $a_1 = 1$ shows that y_2 also satisfies the equation.

Note that the two series defining y_1 and y_2 converges for all x, furthermore, y_1 and y_2 are linearly independent as neither series is a constant multiple of the other.

We therefore see that (0.27) is the general solution of (0.24), and that any particular solution is obtained by specifying the values of $y(0) = a_0$ and $y'(0) = a_1$.

In the above example two series in parentheses are easily recognizable as the expansions of $\cos x$ and $\sin x$, so (0.27) can be written in the form

$$y = a_0 \cos x + a_1 \sin x.$$

We now apply the method of these example to establish the following general theorem about the nature of solutions near ordinary points.

Theorem 0.1.

Let x_0 be an ordinary point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0, \tag{0.28}$$

and let a_0 and a_1 be arbitrary constants. Then there exists a unique function $y(x)$ that is analytic at x_0, is a solution of equation (0.28) in a certain neighborhood of this point, and satisfies the initial conditions $y(x_0) = a_0$ and $y'(x_0) = a_1$. Furthermore, if the power series expansions of $P(x)$ and $Q(x)$ are valid on an interval $|x - x_0| < R$, $R > 0$, then the power series expansion of this solution is also valid on the same interval.

For the proof of this theorem we refer to our text book. We solve the Legendre's equation by the same procedure used to solve the first example.

Legendre's differential equation:

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0, \tag{0.29}$$

where p is a constant. It is clear that the coefficient functions

$$P(x) = \frac{-2x}{1-x^2} \quad \text{and} \quad Q(x) = \frac{p(p+1)}{1-x^2}$$

are analytic at the origin. The origin is therefore an ordinary point and we expect a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Since $y' = \sum (n+1) a_{n+1} x^n$, we get the following expansions for the indvidual terms on the left side of equation (0.29):

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n,$$

$$-x^2y'' = \sum_{n=2}^{\infty} -(n-1)na_nx^n,$$

$$-2xy' = \sum_{n=1}^{\infty} -2na_nx^n,$$

and

$$p(p+1)y = \sum_{n=0}^{\infty} p(p+1)a_nx^n.$$

By equation (0.29), the sum of the these series is required to be zero, so the coefficients of x^n must be zero for every n:

$$(n+1)(n+2)a_{n+2} - (n-1)na_n - 2na_n + p(p+1)a_n = 0.$$

With a little manipulation, this becomes

$$a_{n+2} = -\frac{(p-n)(p+n+1)}{(n+1)(n+2)} a_n.$$

This *recursion formula* enables us to express a_n in terms of a_0 or a_1 according as n is even or odd:

$$a_2 = -\frac{p(p+1)}{1 \cdot 2} a_0,$$

$$a_3 = -\frac{(p-1)(p+2)}{2 \cdot 3} a_1,$$

$$a_4 = -\frac{(p-2)(p+3)}{3 \cdot 4} a_2 = \frac{p(p-2)(p+1)(p+3)}{4!} a_0,$$

$$a_5 = -\frac{(p-3)(p+4)}{4 \cdot 5} a_3 = \frac{(p-1)(p-3)(p+2)(p+4)}{5!} a_1,$$

$$a_6 = -\frac{(p-4)(p+5)}{5 \cdot 6} a_4 = -\frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!} a_0$$

$$a_7 = -\frac{(p-5)(p+6)}{6 \cdot 7} a_5$$
$$= -\frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!} a_1,$$

and so on. By inserting these coefficients into $y = \sum a_n x^n$, we obtain

$$y = a_0 \Big[1 - \frac{p(p+1)}{1 \cdot 2} x^2 + \frac{p(p-2)(p+1)(p+3)}{4!} x^4 - \frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!} x^6 + \cdots \Big] + a_1 \Big[x - \frac{(p-1)(p+2)}{2 \cdot 3} x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!} x^5 - \frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!} x^7 + \cdots \Big] \tag{0.30}$$

as our solution of (0.29).

When p is not an integer, each series in bracket has radius of converges $R = 1$.

This is most easily seen by using the recursion formula: for the first series, this formul (with n replaced by $2n$)

$$\left| \frac{a_{2n+2} x^{2n+2}}{a_{2n} x^{2n}} \right| = \left| \frac{(p-2n)(p+2n+1)}{(2n+1)(2n+2)} \right| |x^2| \to |x^2|$$

as $n \to \infty$, and similarly for the second series.

The fact that the each series has positive radius of convergence justifies the operations we have performed and shows that the power series (0.30) is a valid solution of (0.29) for every choice of constants a_0 and a_1.

Each bracketed series is a particular solution and since it is clear that the functions defined by these series are linearly independent, (0.30) is the general solution of (0.29)

1. Hermite Differential Equation: $y'' - 2xy' + 2py = 0$
2. Laguerre's Differential Equations: $xy'' + (1 - x)y' + py = 0$
3. Chebyshev's equation: $(1 - x^2)y'' - xy + p^2 y = 0'$
4. Airy's Equation: $y'' \pm p^2 xy = 0$

Regular Singular Points:

We recall the point x_0 is a *singular point* of the differentail equation

$$y'' + P(x)y' + Q(x)y = 0, \tag{0.31}$$

if one (or both) of the coeffiecient functions $P(x)$ and $Q(x)$ fails to be analytic at x_0.

In this case the theorem and method of the previous section do not apply. If we wish to study the solution of the above differential equations near x_0, then new ideas are necessary.

At this moment we can not say that the equation has (or has no) a solution near the point x_0.

But if we were in a situation that the coefficient functions are not so far that to be analytic at the point x_0 in the sense a mild modifications of the coefficient functions are analytic, would we be able to find the solutions of equation (0.31)?

The answer is yes partially. We have some singular points called Regular Singular Points near which we can find the solutions.

What are Regular singular points: A singular point x_0 is said to be *regular* if the functions $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytics at x_0. If the singular point is not regular then it is called *irregular*.

Roughly speaking, at regular point x_0, the sigularity of $P(x)$ and $Q(x)$ can not be worse than $1/(x - x_0)$ and $1/(x - x_0)^2$ respectively.
Example:
1. $x^2 y'' + pxy' + rxy = 0$ and 2. $x^3 y'' + px^2 y' + \sin xy = 0$, In both examples the point $x = 0$ is regular point.

3. $x^3(1 - x^2)y'' + 2xy' - 2y = 0$, here $x = 0$ is irregular singular but not regular singular, $x = \pm 1$ are regular singular points and any other point is an ordinary point.

Let p, a, b, c and k are real constants, find out all ordinary points, regular and irregular sigular points of the following equations.

Legender's Differential Equation: $(1 - x^2)y'' - 2xy' + p(p - 1)y = 0$
$x = \pm 1$ are the regular singular point of the above equations all others are ordinary points.

Guass Hypergeometric Differentila Equation:

$$x(x-1)y'' + [c-(a+b+1)x]y' - aby = 0$$

$x = 0, 1$ are the regular singular points and all other points are ordinay points.

Bessel's Differential Equation: $x^2y'' + xy' + (x^2 - p^2)y = 0$
$x = 0$ is only the regular singular point and any other points are ordinay.

Hermite Differential Equation: $y'' - 2xy' + 2py = 0$
There are no singular points. All points are ordinary points.

Laguerre's Differential Equations: $xy'' + (1-x)y' + py = 0$
$x = 0$ is regular singular and others are ordinary point.

How do we solve the differential equation

$$y'' + P(x)y' + Q(x)y = 0, \tag{0.32}$$

near $x = x_0$, if it has regular singularity at $x = x_0$.

In this case we will find a solutions y which is of the form

$$\begin{aligned} y &= (x - x_0)^m[a_0 + a_1(x - x_0)^2 + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots] \\ &= a_0(x - x_0)^m + a_1(x - x_0)^{m+1} + a_2(x - x_0)^{m+2} + a_3(x - x_0)^{m+3} + \cdots \end{aligned} \tag{0.33}$$

where the coefficients $a_0 \neq 0, a_1, a_2, \ldots$ and the exponent m are unknown. The above series is called *Frobenius series*.

We will find the unknown coefficients and the exponent so that y is a solution the equation (0.32), the exponent m may be a negative integer, a fraction or even an irrational real number.

Question: What is the guarantee that the solution of the above form exists?

We answere this question later after we get enough familiar with this kind of solutions. We will see some example to get familiar with those.

We consider the equation

$$2x^2y'' + x(2x+1)y - y = 0. \qquad (0.34)$$

If we write the equation in standard form

$$y'' + \frac{\frac{1}{2}+x}{x}y' + \frac{-1/2}{x^2}y = 0$$

then we see at once that $xP(x) = \frac{1}{2} + x$ and $x^2Q(x) = -\frac{1}{2}$ are analytic at $x = 0$, so $x = 0$ is a regular singular point.

We want to find a solution near this point. We now introduce our assumed *Frobenius series* solution

$$\begin{aligned} y &= x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) \\ &= a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \cdots, \qquad (0.35) \end{aligned}$$

with $a_0 \neq 0$,

and its derivatives are given by

$$y' = a_0 m x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2} + \cdots$$

and

$$y'' = a_0 m(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + a_2(m+2)(m+1)x^m + a_3(m+3)(m+2)x^{m+1} + \cdots .$$

To find the coefficients in (0.35), we proceed in essentially the same way as in the case of an ordinary point, with the significant difference that now we must find the appropriate value (or values) of the exponents m.

When we substitute three series y, y' and y'' in the equation (0.34), then the common factor x^{m-2} of the series y, y'/x and y''/x^2 gets cancel, the result is

$$a_0 m(m-1)+a_1(m+1)mx+a_2(m+2)(m+1)x^2+a_3(m+3)(m+2)x^3+\cdots$$
$$+\left(\frac{1}{2}+x\right)[a_0 m + a_1(m+1)x + a_2(m+2)x^2 + a_3(m+3)x^3 + \cdots]$$
$$-\frac{1}{2}[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots] = 0$$

By inspection, we combine correspoding powers of x and equate the coefficient of each power of x to zero.

This yields the following system of equations:

$$\begin{aligned} a_0\left[m(m-1)+\frac{1}{2}m-\frac{1}{2}\right] &= 0, \\ a_1\left[(m+1)m+\frac{1}{2}(m+1)-\frac{1}{2}\right]+a_0 m &= 0, \\ a_2\left[(m+2)(m+1)+\frac{1}{2}(m+2)-\frac{1}{2}\right]+a_1(m+1) &= 0, \\ &\cdots . \end{aligned} \tag{0.36}$$

Since $a_0 \neq 0$, it follows from first of the above equations that

$$m(m-1) + \frac{1}{2}m - \frac{1}{2} = 0.$$

This is called the *indicial equation* of the differential euqation (0.34). Its roots are

$$m_1 = 1 \qquad \text{and} \qquad m_2 = -\frac{1}{2}$$

These are only possible values for the exponent m in (0.35).

For each of these values of m, we now use the remaining equations of (0.36) to calculate $a_1, a_2, a_3, \ldots$ in terms of a_0. For $m_1 = 1$, we obtain

$$a_1 = -\frac{a_0}{2 \cdot 1 + \frac{1}{2} \cdot 2 - \frac{1}{2}} = -\frac{2}{5}a_0$$

$$a_2 = -\frac{2a_1}{3 \cdot 2 + \frac{1}{2} \cdot 3 - \frac{1}{2}} = -\frac{2}{7}a_1 = \frac{4}{35}a_0,$$

$$\cdots.$$

And for $m = -\frac{1}{2}$, we obtain

$$a_1 = \frac{\frac{1}{2}a_0}{\frac{1}{2}\left(-\frac{1}{2}\right) + \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2}} = -a_0$$

$$a_2 = -\frac{\frac{1}{2}a_1}{\frac{3}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{3}{2} - \frac{1}{2}} = -\frac{1}{2}a_1 = \frac{1}{2}a_0,$$

$$\cdots.$$

We therefore have the following two Frobenius series solutions in each of which we have put $a_0 = 1$:

$$y_1 = x\left(1 - \frac{2}{5}x + \frac{4}{35}x^2 + \cdots\right) \tag{0.37}$$

$$y_2 = x^{-1/2}\left(1 - x + \frac{1}{2}x^2 + \cdots\right). \tag{0.38}$$

These solutions are clearly linear independent for $x > 0$, so the general solution of (0.34) on this interval is

$$y = c_1x\left(1 - \frac{2}{5}x + \frac{4}{35}x^2 + \cdots\right) + c_2x^{-1/2}\left(1 - x + \frac{1}{2}x^2 + \cdots\right)$$

In view of the above example, it is easy to see that the indicial equation of the more general diffierential equation (0.32) is

$$m(m-1) + mp_0 + q_0 = 0. \tag{0.39}$$

where p_0 and q_0 are constant terms of the power series expansions of $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ near the point $x = x_0$ respectively.

In our example, the inditial equation had two distinct real roots leading to the two independent series solutions (0.37) and (0.38).

It is natural to expect such a result whenever the indicial equation has distinct real roots m_1 and m_2. This turns out to be true if the difference between m_1 and m_2 is not an integer.

If, however, this difference is an integer, then it often (but not always) happens that one of two expected series solution does not exist.

In this case it is necessary just as in the case $m_1 = m_2$ to find a second solution by other methods. In the next section we investigate these difficulties in greater detaills.

Thank you for your attention

Happy Weekend !