Ch. 6: Angular Momentum and Fixed Axis Rotation

R I S H I K E S H   V A I D Y A
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What is Rotation?

1-Dim World: Concept of direction (and hence vectors) meaningless. Magnitudes sufficiently characterize physical quantities. A change is only a change in magnitude.

2-Dim World: Infinitely many directions and hence the concept of vectors. Vectors can change in two ways.

- Pure Scaling: Changes in magnitude only (Δr)
- Pure Rotation: Changes in direction only ($\Delta\theta$)

Rotation quantifies pure change in direction

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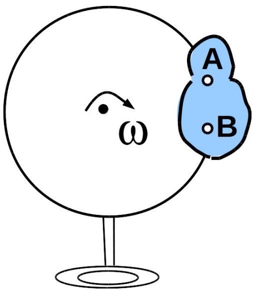
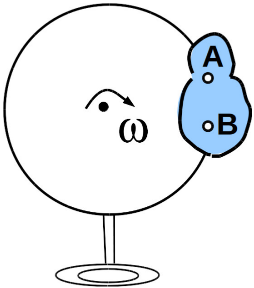
A rigid body suspended on a rotating disk



- Does the suspended rigid body rotate when hooked only from one frictionless pivot at A?
- Does it rotate if is in addition hooked at another point say B.?

We must first make a distinction between point particles and extended rigid bodies and then define what does rotation and translation mean for a rigid body.

A rigid body suspended on a rotating disk



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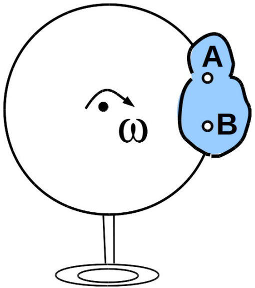

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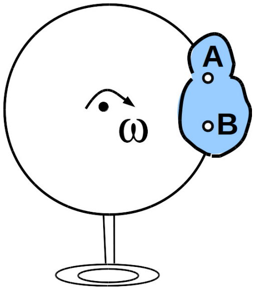

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Rigid bodies are a different ‘breed’ from point particles

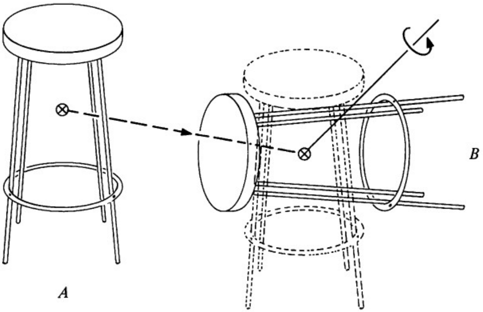
So far we have ignored size, shape, and extensions of bodies. We could get away with it as we were only interested in gross translational motion of objects as we dumped all of the mass (no matter how it was distributed) at a point called center of mass and worked out its translation.

A rigid body is one for which the distance between any two of its particles remains constant. So essentially it is an idealized solid that does not undergo deformation. A rigid body is capable of far more complicated motion

Rigid bodies are a different ‘breed’ from point particles

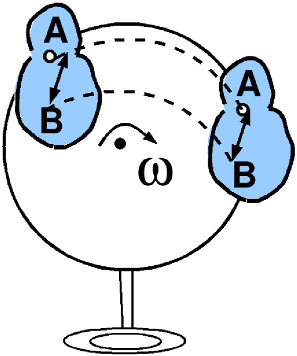
Chasles’ theorem: Any displacement of a rigid body can be decomposed into two independent motions: a translation of the center of mass and a rotation about the center of mass.

Rigid Body Motion = Translation of C.M. + Rotation about C.M



A point particle abstraction being devoid of extension, the notion of rotation about its CM of is meaningless.

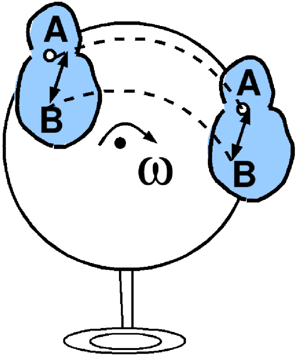
When does a rigid body translate?



Translation: The line connecting any two particles of body retains its direction in space.

Rotation: The trajectories of all points of the body are circles whose centers lie in a common straight line called the axis of rotation.

When does a rigid body translate?



When hooked only at A, it undergoes translation, but when hooked at any two points, (say A and B), it undergoes rotation about an axis through the center of the disk.

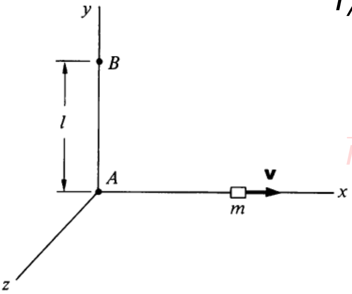
Since we must distinguish between motions of point particles and rigid bodies, let us begin with the idiosyncrasies of the angular momentum $\vec{L}$ of a point particle.

$\vec{L}$ of a point particle is a subjective notion!

$$\vec{L} = \vec{r} \times \vec{p}$$

Since the definition of $\vec{L}$ involves $\vec{r}$ which depends on choice of origin, $\vec{L}$ also depends on choice of origin. Period.

Anuglar momentum of a sliding block about point A and B is different



$$\vec{r}_A = x\hat{i}, \quad \vec{v} = v\hat{i}$$

$$\vec{L}_A = \vec{r}_A \times m\vec{v} = 0$$

$$\vec{r}_B = x\hat{i} - l\hat{j}$$

$$\vec{L}_B = \vec{r}_B \times m\vec{v} = mlv\hat{k}$$

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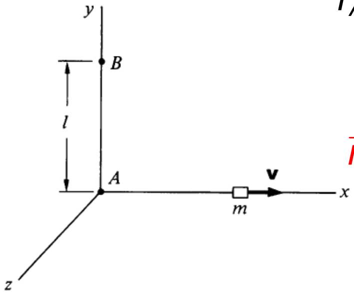

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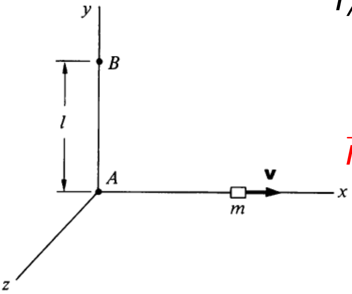

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Okay, So you are essentially saying that the spin of Earth is subjective!

So how do I convince my friend in the US that the rotation of the earth is an illusion owing to a choice of origin and so is the perception of day and night and jet-lag and all that ?

Okay, So you are essentially saying that the spin of Earth is subjective!

No, I am not saying that. You must make a distinction between the orbital angular momentum of a rigid body (measured with respect to any arbitrary origin other than the center of mass of the rigid body) and the spin angular momentum, which is by definition, the angular momentum of a rigid body in it's center of mass frame. The former (orbital) is a function of the choice of origin but the latter (spin) is unique, being measured by definition, in the center of mass frame. Even for a two body system, similar considerations apply.

Having defined $\vec{L}$ for a particle, let us move on to understanding $\vec{L}$ for a rigid body. As the title of this chapter suggests, we shall be restricting our attention to the case fixed axis rotation of a rigid body.

Why a fixed axis rotation?

A general problem of rigid body motion where a body is free to rotate about any axis is certainly more realistic and interesting, however, mathematically it is significantly more involved. A better understanding of fixed axis rotation is a pre-requisite for this.

This means that the axis of rotation can sure translate, maintaining its direction (like the wheel of a car going in a straight line) but cannot rotate (as in the car taking a turn).

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Can I write: $\vec{\theta} = \theta_x \hat{i} + \theta_y \hat{j} + \theta_z \hat{k}$

No! A rotation by a finite angle cannot be written as a vector.

What is the problem? From where does $\vec{\omega} = d\vec{\theta}/dt$) inherit its vector character if not from $\vec{\theta}$?

For physical quantities $\vec{A}_1$ and $\vec{A}_2$ to qualify as a vector: $\vec{A}_1 + \vec{A}_2 = \vec{A}_2 + \vec{A}_1$. That is vector addition is commutative. This is not true for two rotations by finite angles θ. However, it turns out that rotation by infinitesimal angles is commutative and hence $d\vec{\theta}$ is a vector and that is all we need to define another vector $\vec{\omega}$.

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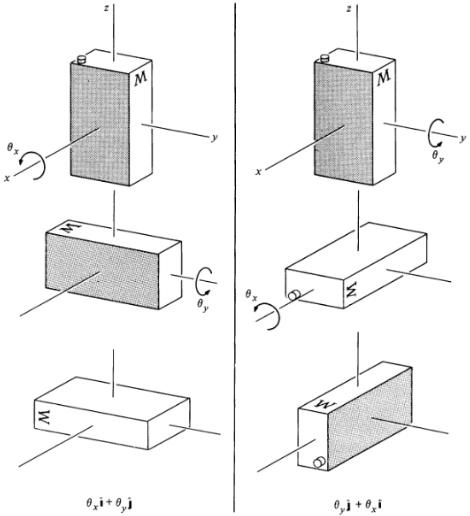
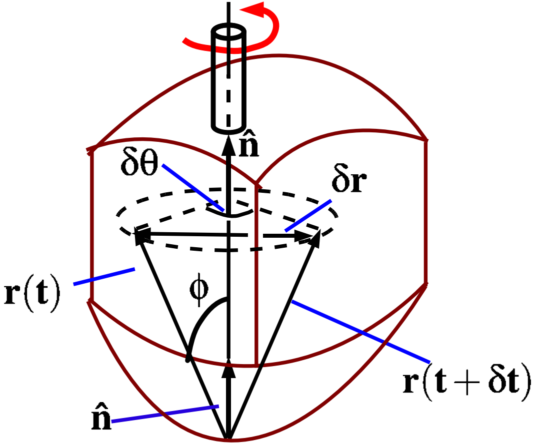
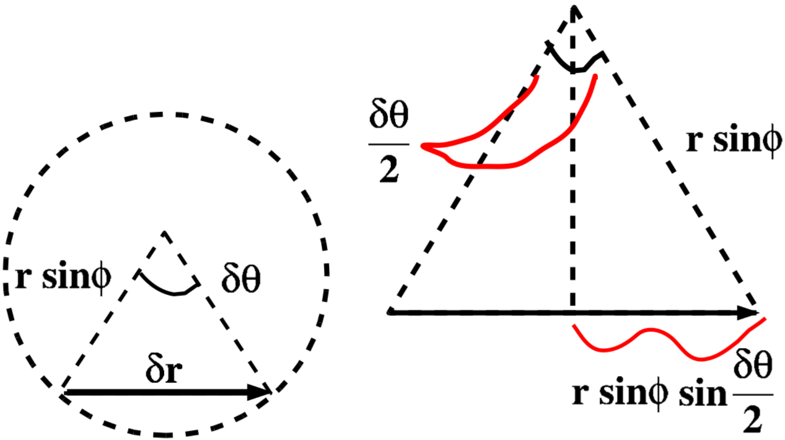
Finite rotations do not commute: Here is the proof

$$\theta_x\hat{i} + \theta_y\hat{j} \neq \theta_y\hat{j} + \theta_x\hat{i}$$

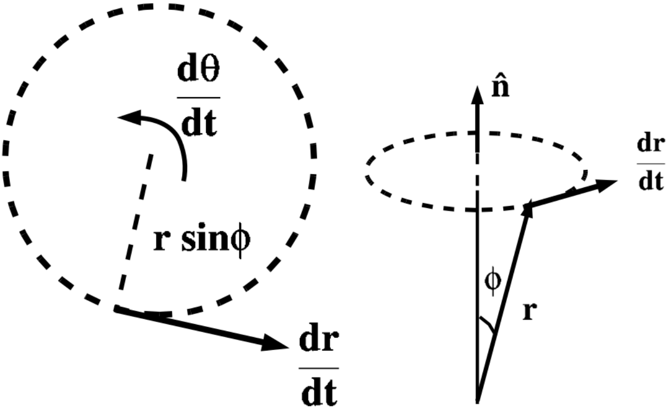




Consider a rigid body rotating about some axis and let $\hat{n}$ be the instantaneous axis of rotation. We want to find the relation between $\vec{r}$, $\vec{v}$, $\vec{\omega}$ for a particle at position vector $\vec{r}$ as shown.


r sinϕ
δθ
δr
δθ/2
r sinϕ
r sinϕ sin δθ/2

Relation between $\vec{r}$, $\vec{v}$, $\vec{\omega}$



$$\vec{v} = \vec{\omega} \times \vec{r}$$

Work Energy Theorem for Rigid Body Translating and Rotating

When a rigid body goes from point *a* to *b*, let the work done be W_{ba}.

$$w_{ba} = K_b - K_a$$

Since W_{ba} is total work done due to the force responsible for translation and torque responsible for fixed axis rotation:

$$\begin{aligned} W_{ba} &= \int_a^b \vec{F} \cdot d\vec{R} + \int_{\theta_a}^{\theta_b} \tau_o d\theta \\ &= \left(\frac{1}{2}MV_b^2 - \frac{1}{2}MV_a^2\right) + \left(\frac{1}{2}I_0\omega_b^2 - \frac{1}{2}I_0\omega_a^2\right) \end{aligned}$$

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The Force(torque)-Work-Energy Triangle

$$\vec{F} = m\frac{\vec{dv}}{dt}$$

The Force(torque)-Work-Energy Triangle

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

Integrate with respect to x

→

Work Energy Theorem

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

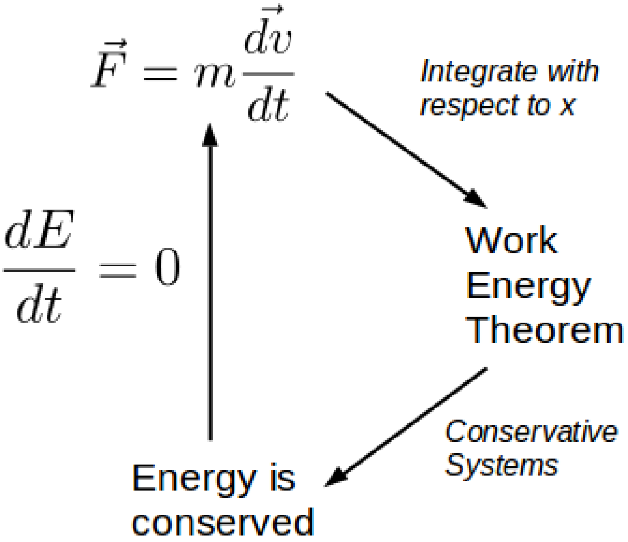
Integrate with respect to x

Work Energy Theorem

Conservative Systems

Energy is conserved

The Force(torque)-Work-Energy Triangle

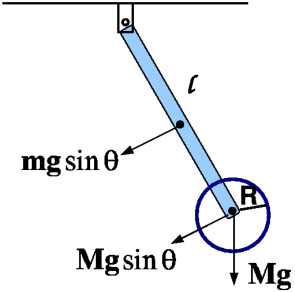


Its all about 'Moments' of Inertia

A poem on lessons learnt from problem 6.18 on M.I. (circa 2009)

Yesterday I made the disk spin
For it was free to spin
But today I retracted –
Free to spin also means
Free not to spin
And like all the choices
between exerting and not

The disk chooses not to spin.



Its all about ‘Moments’ of Inertia

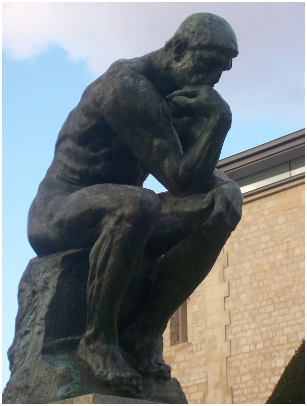
A poem on lessons learnt from problem 6.18 on M.I. (circa 2009)

Reality has no obligation
To the sweat of our cerebration
Nor does it give a damn
to a certain Newton’s conviction
She goes about her song and dance
Which should you care to hear and see
Go with hunger and humility
A sac full of lifetime’s patience
But not a shred of expectation
For she has no obligation
to reward your cerebral salination.

Its all about 'Moments' of Inertia

A poem on lessons learnt from problem 6.18 on M.I. (circa 2009)

Physics-1 can be Fun
Only if you think and question
But before you ask one
Exercise caution
Solution borrowed is no fun
The discovered one
is whats yours in the long run.
Not saying don't question
Just think before you ask one.



Its all about 'Moments' of Inertia

A poem on lessons learnt from problem 6.18 on M.I. (circa 2009)

Its cool as well as hot
Only if you give it a shot
Why the lady with a pot
sways a lot?
It is c.g. (without p.a.)
that matters a lot
Only if you ask why & why not
Your c.g. (along with p.a.)
Will rise a lot.

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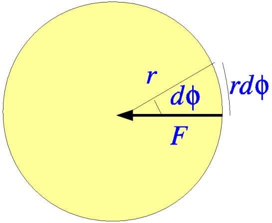
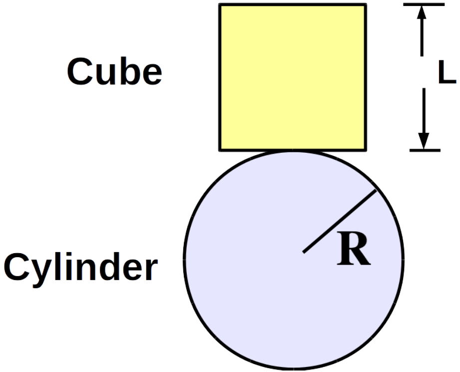
Its all about 'Moments' of Inertia

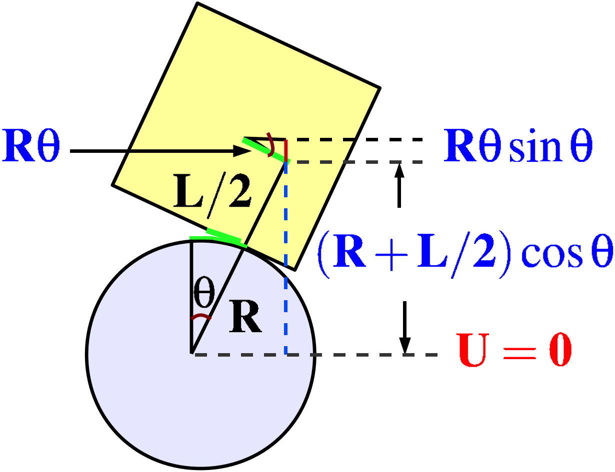
A poem on lessons learnt from problem 6.18 on M.I. (circa 2009)

It is tastier than french-fry
Only if you don't sit and cry
Take a problem and give it a try
Its not just force, but the length of pry
If your all F's along dr lie
Your efforts shall surely fructify
If F is small, just stay put to impulsify.
But if your dr is R dϕ
Normal to all F_i
You are destined to cry.

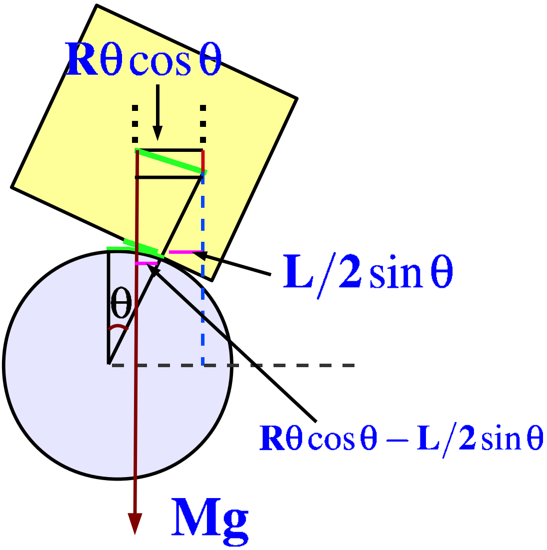


A cubicle block of side L rests on a fixed cylindrical drum of radius R. Find the largest value of L for which the box is stable.



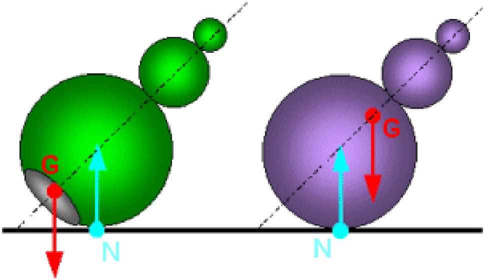


$$h = R\theta\sin\theta + (R+L/2)\cos\theta$$



Rθ cos θ
L/2 sin θ
θ
Rθ cos θ − L/2 sin θ
Mg

Prob. 6.35



Ch. 8: Non Inertial Systems and Fictitious Forces

R I S H I K E S H   V A I D Y A

Theoretical Particle Physics

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Principle of Equivalence

There is no way to distinguish locally between a uniform gravitational acceleration $\vec{g}$ and an acceleration of the coordinate system $\vec{A} = -\vec{g}$. Locally here means a sufficiently small region like that of an elevator) where you can assume $\vec{g}$ to be practically constant.

For instance if your elevator is in a state of free fall under earth's gravity and you drop an apple, then the apple will float in front of you($m_i = m_g$)

$$\vec{F}' = \underbrace{-m_g\vec{g}}_{real\ \ force} + \underbrace{m_i\vec{g}}_{Fictitious\ \ force} = 0$$

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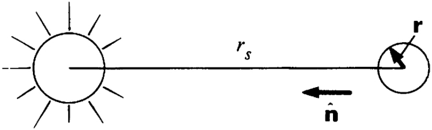
That means inside of a sufficiently giant elevator since $\vec{g}$ would vary from point to point, I can see departures from this equivalence as cancellation won't be exact.

Earth as a giant elevator

Earth is in a state of free-fall towards Sun.



$$\vec{G}_0 = G\,M_s \frac{\hat{n}}{r_s^2}$$

If $\vec{G}(\vec{r})$ is the gravitational field of Sun at some location $\vec{r}$ on earth. Then,

$$\vec{F} = m\,\vec{G}(\vec{r})$$

The apparent force to an earthbound observer is

$$\vec{F}' = F - m\,\vec{A} = m\,[\vec{G}(\vec{r}) - \vec{G}_0] = m\,\vec{G}'(\vec{r})$$

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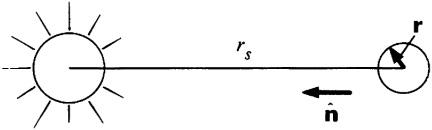

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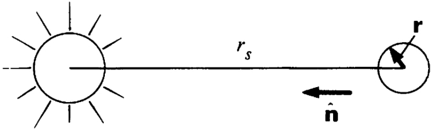

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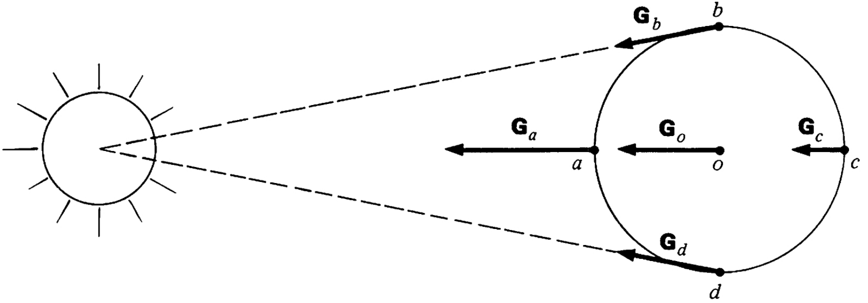
Sun's gravitational field $\vec{G}(\vec{r})$ at different points on Earth



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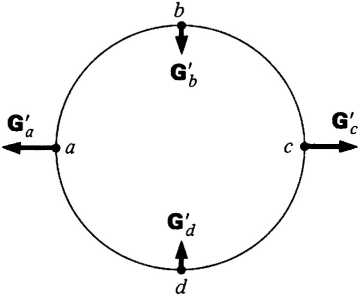
Apparent field $\vec{G}'(\vec{r})$ at different points on earth's surface



$$G_a = \frac{G\,M_s}{(r_s - R_e)^2}$$

Calculation of G'_a and G'_c:

$$\begin{aligned} G'_a &= G_a - G_0 \\ &= \frac{G\,M_s}{(r_s - R_e)^2} - \frac{G\,M_s}{r_s^2} \\ &= \frac{G\,M_s}{r_s^2}\left[\frac{1}{[1-(R_e/r_s)^2]} - 1\right] \end{aligned}$$

Apparent field $\vec{G}'(\vec{r})$ at different points on earth's surface

Since $\frac{R_e}{r_s} = \frac{6.4\times10^3 \ km}{1.5\times10^8 \ km} = 4.3\times10^{-3} << 1$, we have

$$
\begin{aligned}
G'_a &= G_0\left[\left(1-\frac{R_e}{r_s}\right)^{-2}-1\right] \\
&= G_0\left[1+2\frac{R_e}{r_s}+\ldots-1\right] \\
&= 2\,G_0\frac{R_e}{r_s}
\end{aligned}
$$

where we have neglected terms of order $(R_e/r_s)^2$ and higher.

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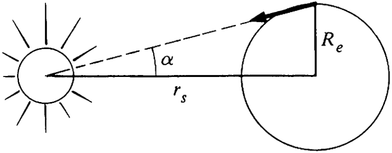
Apparent field $\vec{G}'(\vec{r})$ at different points on earth's surface

Similarly for point c, the distance from the Sun changes to $r_s + R_e$. Hence,

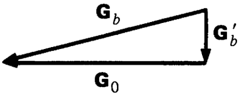
$$G'_c = -2\,G_0 \frac{R_e}{r_s}$$

Apparent field $\vec{G}'(\vec{r})$ at different points on earth's surface

Calculation of G'_b and G'_d:



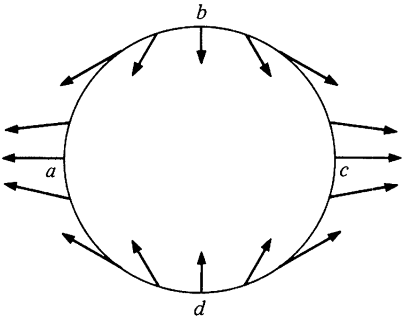
Points b and d are approximately same distance from the Sun as the center of the earth. However, $\alpha \approx R_e/r_s = 4.3 \times 10^{-5} << 1$.



$$
\begin{aligned}
G'_b &\approx G_0\,\alpha \\
&\approx G_0 \frac{R_e}{r_s}
\end{aligned}
$$

By Symmetry G'_d is equal but opposite to G'_b.

Apparent field $\vec{G}'(\vec{r})$ at different points on earth's surface



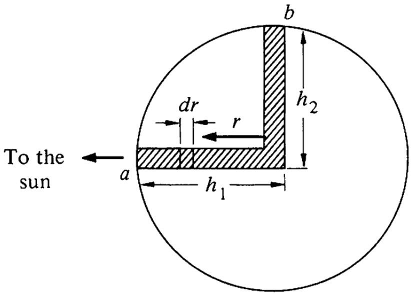
Tidal Forces

Forces at *a* and *c* tend to lift the ocean

Forces at *b* and *d* tend to depress them.

Can you see why there are two tides in a day?

Equilibrium Height of a Tide: Newton's Model



Two orthogonal wells

Pressure due to short column of water of height dr is $\rho g(r)dr$ where $g(r)$ is the effective gravitational field at r. For equilibrium:

$$\int_0^{h_1} \rho\, g_1(r)dr = \int_0^{h_2} \rho\, g_2(r)dr$$

The idea is to calculate $\Delta h = h_1 - h_2$, the height of tide due to Sun.

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Effective field toward the center of earth in column 1:

$$g_1(r) = \underbrace{g(r)}_{Earth's\ gravitational\ field} - \underbrace{G_1'(r)}_{Apparent\ field\ of\ Sun}$$

Borrowing from G_a' (substitute r for R_e):

$$G_1'(r) = \frac{2\,G\,M_s\,r}{r_s^3} = 2\,C\,r$$

where $C = G\,M_s/r_s^3$.

Thus,

$$\begin{aligned} g_1(r) &= g(r) - 2\,C\,r \\ g_2(r) &= g(r) + G_2'(r) \\ &= g(r) + C\,r \end{aligned}$$

Condition for equilibrium is:

$$\begin{aligned} \int_0^{h_1} [g(r) - 2\,c\,r]dr &= \int_0^{h_2} [g(r) \; + C\,r]dr \\ \int_0^{h_1} g(r)dr - \int_0^{h_2} [g_(r)dr &= \int_0^{h_1} 2\,C\,rdr + \int_0^{h_2} C\,r\,dr \\ \int_{h_2}^{h_1} g(r)dr &= \int_0^{h_1} 2\,C\,rdr + \int_0^{h_2} C\,r\,dr \end{aligned}$$

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Since $h_1 \approx h_2 \approx R_e$, $g(r) \approx g(R_e) = g$
above equation reduces to

$$g\Delta h_s = \frac{3}{2} C R_e^2$$

$$\Delta h_s = \frac{3}{2}\frac{M_s}{M_e}\left(\frac{R_e}{r_s}\right)^3 R_e \quad \left[g = \frac{GM_s}{R_e^2} \quad C = \frac{GM_s}{r_s^3}\right]$$

Using the data

$M_s = 1.99 \times 10^{33}$ g $\qquad r_s = 1.49 \times 10^{13}$ cm

$M_e = 5.98 \times 10^{27}$ g $\qquad R_e = 6.37 \times 10^{8}$ cm,

we obtain

$$\Delta h_s = 24.0 cm$$

An identical calculation for moon yields:

$$\Delta h_m = \frac{3}{2}\frac{M_m}{M_e}\left(\frac{R_e}{r_m}\right)^3 R_e = 53.5\,cm$$

Since $\Delta h \to 1/r^2$, the distance factor more than kills whatever advantage Sun has due to its mass. Strongest tides (spring tides) occur when moon and Sun act along the same line. Weak (neap tides) occur midway between, at the quarters of the moon.

$$\frac{\Delta h_{spring}}{\Delta h_{neap}} = \frac{\Delta h_m + \Delta h_s}{\Delta h_m - \Delta h_s} \approx 3$$

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