Reliability Theory

- ✓ What is reliability?
- ✓ Why it is important to study?
- ✓ Use of reliability in day to day life.

Reliability may be defined in several ways:

- ➢The idea that an item is fit for a purpose with respect to time.
- ➢ In the most discrete and practical sense: "Items that do not fail in use are reliable" and "Items that do fail in use are not reliable”.

Reliability of a device is denoted as $R(t)$ and defined as

$$R(t) = P\left[T > t\right],\ t > 0.$$

Where "T" is a continuous random variable representing the life length of the device.

Some property of the reliability function $R(t)$:

$$(i) \quad R(t) = \int_t^{\infty} f(u)du$$

Where "f" is probability density function (p.d.f) of "T".

We assume

$$f(t) = 0 \ for \ t < 0$$

$$(ii) \quad R'(t) = -f(t)$$

(iii) $R(t)$ is a nonincreasing function of t

(iv) $R(0) = 1$, $R(\infty) = 0$

(v) $R(t) = 1 - F(t)$ where $F(t)$ is the cdf of T

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$$R(t) = P(T > t)$$

$$or,\ R(t) = 1 - P(T \le t)$$

$$or,\ R(t) = 1 - F(t)$$

Hence the results

(iv) $R(0) = 1$, $R(\infty) = 0$

$(v) \Rightarrow (iv)$

$(iii)\;\; R(t)$ is a nonincreasing function of t

$(iv) \Rightarrow (iii)$

$(ii)\;\; R'(t) = -f(t)$

$(v) \Rightarrow (ii)$

$(v) \Rightarrow R(t) = 1 - F(t)$

$\Rightarrow R'(t) = -F'(t) = -f(t)$

Hence the results

Conditional failure (hazard) rate function:

It is denoted by $Z(t)$ *or* $h(t)$

$$Z(t) = \lim_{\Delta t \to 0} \frac{P\left[t < T \leq t + \Delta t / T > t\right]}{\Delta t}$$

For small Δt

$$Z(t)\Delta t \cong P\left[t < T \leq t + \Delta t / T > t\right]$$

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For small $\Delta t, Z(t)\Delta t$ approximates the conditional probability that the device will fail during the interval $\left(t, t+\Delta t\right)$ given that it has survived at the time t

In particular, if $Z(t) = \lambda$

$\Rightarrow$ Hazard rate is constant

In that case we can say that the device is as good as new at any time of its life.

Exercise: *Show that* $Z(t) = \dfrac{f(t)}{R(t)}$.

$$Z(t) = \lim_{\Delta t \to 0} \frac{P\left(t < T < t + \Delta t / T > t\right)}{\Delta t}$$

$$Z(t) = \lim_{\Delta t \to 0} \frac{P\left[\left(t < T < t + \Delta t\right) \bigcap \left(T > t\right)\right]}{\Delta t . P\left(T > t\right)}$$

$$Z(t) = \lim_{\Delta t \to 0} \frac{P\left[\left(t < T < t + \Delta t\right)\right]}{\Delta t . P\left(T > t\right)}$$

$$Z(t) = \lim_{\Delta t \to 0} \frac{\int_{t}^{t+\Delta t} f(u)du}{\Delta t.R(t)}$$

$$Z(t) = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t.R(t)}$$

$$Z(t) = \frac{F'(t)}{R(t)} = \frac{f(t)}{R(t)}.$$

Exercise :

Find the failure rate function for Weibull Distribution:

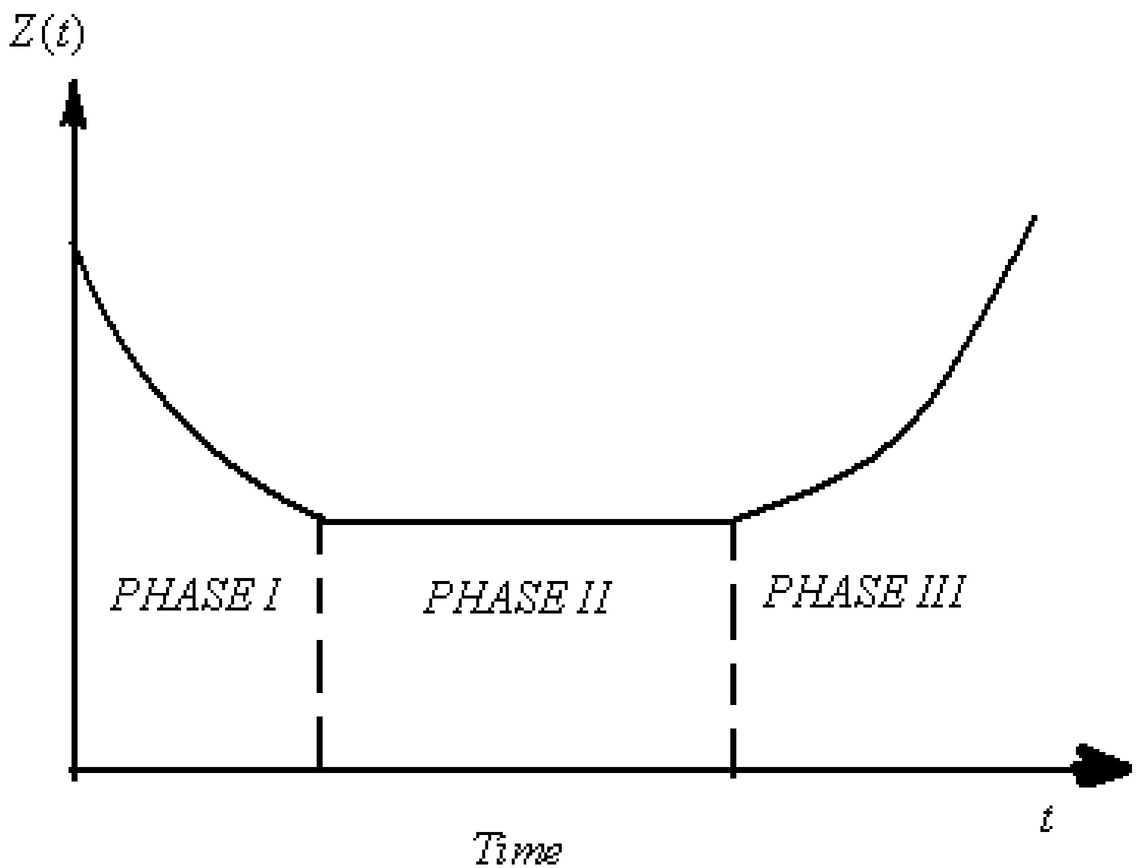
$$F(t) = 1 - e^{-(\lambda t)^{\alpha}}$$

$$\Rightarrow Z(t) = \frac{f(t)}{R(t)} \Rightarrow Z(t) = \frac{F'(t)}{1 - F(t)}$$

$$F'(t) = e^{-(\lambda t)^{\alpha}} \alpha (\lambda t)^{\alpha - 1} \lambda$$

$$\Rightarrow Z(t) = \frac{e^{-(\lambda t)^{\alpha}} \alpha (\lambda t)^{\alpha - 1} \lambda}{e^{-(\lambda t)^{\alpha}}} = \alpha \lambda (\lambda t)^{\alpha - 1}$$

Failure Patterns



Theorem:

If *T* is the time to failure of a device with pdf *f(t)* and cdf *F*, and *Z(t)* is the failure rate function of the device, then :

$$R(t) = e^{-\int_0^t Z(s)\,ds}, \quad f(t) = Z(t)e^{-\int_0^t Z(s)\,ds}, \text{ for } t > 0.$$

$$Z(s) = \frac{f(s)}{R(s)} = \frac{F'(s)}{R(s)} = -\frac{R'(s)}{R(s)}$$

Integrating, Note that $R(0) = 1$

$$\int_0^t Z(s)ds == -\int_o^t \frac{R'(s)}{R(s)}ds$$

$$-\ln\left(R(t)\right) = \int_o^t Z(s)ds \Rightarrow R(t) = e^{-\int_0^t Z(s)ds}$$

$$Z(t) = \frac{f(t)}{R(t)}$$

$$f(t) = Z(t)R(t)$$

$$f(t) = Z(t)e^{-\int_0^t Z(s)\,ds}$$

Theorem 6.3:

Expected system life time: Theorem 6.3 shows how mean/average lifetime of a system can be determined from a knowledge of the reliability function R(t)

$$E(T) = \int_0^{\infty} R(s)ds$$

Where T is the time to failure

To prove that, it is sufficient to show that

$$E(T) = \int_0^{\infty} P(T > t)\,dt$$

$$R.H.S = \int_0^{\infty} \left[\int_t^{\infty} f(u)\,du \right] dt$$

$$= \int_0^{\infty} \left[\int_0^{u} f(u)\,dt \right] du$$

$$= \int_0^{\infty} u f(u)\,du = L.H.S$$

Example:

$$F(t) = t/10 \quad 0 \le t \le 10$$
$$= 1 \quad t > 10$$
$$Find \;\; E(T)$$

$$Clealy:$$
$$R(t) = 1 - F(t)$$
$$= 1 - t/10 \quad 0 \le t \le 10$$
$$= 0 \quad t > 0$$

$$E(T) = \int_0^{\infty} R(s)ds$$

$$E(T) = \int_0^{10} \left(1 - \frac{s}{10}\right) ds$$

$$E(T) = \left(s - \frac{s^2}{20}\right)\Bigg|_0^{10} = 10 - 5 = 5$$

Reliability of a system

- Series System:- A system whose components(say n components) are arranged in such a way that the system fails whenever any of its components fail is called a series system.

$$R_s = \prod_{i=1}^{n} R_i$$

$$where \quad R_s \rightarrow \text{reliability of series system}$$

$$R_i \rightarrow \text{Reliability of the ith component}$$

Parallel System

- A system whose components are arranged in such a way that the system fails only if all its components fail is called a parallel system

$$R_s(t) = 1 - \prod_{i=1}^{n} (1 - R_i(t))$$

where $R_s \rightarrow$ reliability of parallel system

$R_i \rightarrow$ Reliability of ith component of the system

Example 1: Consider a series system of three independent components and the life length of each component follows uniformly distribution over (0,10).

(i) Find the reliability of the system.

(ii) Find the expected system life.

Solution: $R_s(t) = \prod_{i=1}^{3} R_i(t) = \prod_{i=1}^{3} \left(1 - F_i(t)\right)$

$$F_i(t) = t/10 \quad 0 \le t \le 10$$
$$= 1 \qquad t > 10$$

$$R_s(t) = \left(1 - F_i(t)\right)^3$$
$$= \left(1 - t/10\right)^3 \quad 0 \le t \le 10$$
$$= 0 \qquad t > 10$$

$$E(T) = \int_{0}^{\infty} R(s)ds$$

$$E(T) = \int_{0}^{10} \left(1 - \frac{s}{10}\right)^{3} ds$$

$$E(T) = \frac{5}{2}$$

Example 2:

Consider a system consisting of eight components . The system consists of five assemblies in series, where assembly 1 consists of component 1 with reliability 0.99, assembly II consists component 2 and 3 in parallel with reliabilities 0.95 and 0.95, assembly III consists component 4,5,6 in parallel with reliabilities 0.96, 0.92, 0.85, assembly IV and V consist of component 7 and 8 with reliabilities 0.95, 0.82 respectively. calculate system reliability.

Exercise VI.7 :

Let Y be the life of an independent series system with n component. Find the pdf and cdf of Y. Also prove that the hazard rate function of the system is the sum of the hazard functions of its components.

Solution:

$$R_s(t) = \prod_{i=1}^{n} R_i(t)$$

$$F_Y(t) = 1 - R_s(t) = 1 - \prod_{i=1}^{n} R_i(t)$$

$$f_Y(t) = -R_s'(t) = -\frac{d}{dt}\left(\prod_{i=1}^{n} R_i(t)\right)$$

$$or,\ \ f_Y(t) = -\sum_{i=1}^{n}\left(R_i'(t)\prod_{\substack{j=1\\ j\neq i}}^{n} R_j(t)\right)$$

$$Z_s(t) = \frac{f_Y(t)}{R_s(t)} = \frac{-\sum_{i=1}^{n}\left(R_i'(t)\prod_{\substack{j=1 \\ j\neq i}}^{n} R_j(t)\right)}{\prod_{i=1}^{n} R_i(t)}$$

$$Z_s(t) = -\frac{R_1'(t)}{R_1(t)} - \frac{R_2'(t)}{R_2(t)} - \cdots\cdots - \frac{R_n'(t)}{R_n(t)}$$

$$Z_s(t) = \frac{f_1(t)}{R_1(t)} + \frac{f_2(t)}{R_2(t)} + \cdots\cdots + \frac{f_n(t)}{R_n(t)}$$

$$Z_s(t) = Z_1(t) + Z_2(t) + \cdots\cdots + Z_n(t)$$

$$Z_s(t) = \sum_{i=1}^{n} Z_i(t)$$

Exercise VI.8.

Consider an n-component independent series system. Given that these components have constant failure rates $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively. Find the

(i) reliability of the system.

(ii) hazard rate of the system.

(iii) pdf of the life-length of the system.

Also express the mean life of the system in terms of mean life of its components.

Solution:

$$R_s(t) = \prod_{i=1}^{n} R_i(t) = \prod_{i=1}^{n} \left(1 - F_i(t)\right)$$

$$R_s(t) = \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-\left(\sum_{i=1}^{n} \lambda_i\right) t}$$

$$Z(t) = Z_1(t) + Z_2(t) + \cdots\cdots + Z_n(t)$$

$$= \lambda_1 + \lambda_2 + \cdots\cdots + \lambda_n$$

$$f_s(t) = -\frac{d}{dt}(R_s(t)) = \left(\sum_{i=1}^{n} \lambda_i\right) e^{-\left(\sum_{i=1}^{n} \lambda_i\right) t}$$

The mean life length of the system is

$$E(T) = \mu_s = \frac{1}{\lambda_1 + \lambda_2 + \cdots + \lambda_n}$$

$$= \frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2} + \cdots + \frac{1}{\mu_n}}$$

where μ_i is the mean life-length of i^{th} component.

Exercise VI.9.

Let T be the life length of an independent parallel system with n-components.

(i) Find the pdf and cdf of T

(ii) Show that

$$e^{-\int_0^t Z(s)ds} = 1 - \prod_{i=1}^{n}\left(1 - e^{-\int_0^t Z_i(s)ds}\right)$$

Solution:

Hint: Use properties of reliability, f(t) and their definitions.

Example:

Probability that a device can survive after 0,1,2,3,4 or more shocks are 1,0.8,0.4,0.2, 0 respectively. If the arrival of shocks follow Poisson distribution with λ=0.15, find R(10).

Solution:

$$R(t) = e^{-\lambda t}\left[1 + 0.8(\lambda t) + 0.4\frac{(\lambda t)^2}{2} + 0.2\frac{(\lambda t)^3}{6}\right]$$

Put λ=0.15 and t=10 in the above.

CERTAIN LIFE MODELS

Exponential: In reliability, this distribution plays a vital role. This is the distribution during phase II.

“**Useful life phases”** of the device, when failure occurs due to an external causes, which may be called **“Shock/shocks”**

Suppose that a device fails due to shocks, which occur in a Poisson process with a rate λ. Assume that the device fails as soon as the first shock arrives. Then:

$$F(t) = 1 - e^{-\lambda t} \;,\; t > 0$$

$or,$

$$f(t) = \lambda e^{-\lambda t} \;,\; t > 0$$
$$= 0 \; else \; where$$

Thus the failure-time distribution is exponential distribution.

Let us assume, "*p*" is probability that even after the shock, no failure occurs.

$$R(t) = P\left[\text{No failure occurs in the time interval } (0,t]\right]$$

$$R(t) = \sum_{n=0}^{\infty} P\left[n \text{ shocks arrive in time } (0,\text{t}] \bigcap \text{device survives all } n \text{ shocks}\right]$$

Using:

$$P(A \bigcap B) = P(A)P\left(B/A\right)$$

$$R(t) = \sum_{n=0}^{\infty} P\left[X_t = n\right] P\left[\text{The device will survives all } n \text{ shocks} \Big/ X_t = n\right]$$

$$R(t)=\sum_{n=0}^{\infty}P\left[X_t=n\right]p^n$$

Where X_t is the number of shocks which arrives in the time-interval $(0,t]$. Under the assumption that the shocks arrive in $(0,t]$ is a Poisson process, then X_t has a Poisson distribution with parameter λt

$$R(t)=\sum_{n=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^n}{n!}p^n, \quad t>0$$

$$R(t)=e^{-\lambda t}\sum_{n=0}^{\infty}\frac{\left(p\lambda t\right)^n}{n!} \quad \Rightarrow R(t)=e^{-\lambda(1-p)t} \text{..........}(*)$$

Differentiate (*) with respect to t we get

$$\Rightarrow R(t) = e^{-\lambda(1-p)t} \text{...........} (*)$$

$$R'(t) = -f(t) = -\lambda(1-p)e^{-\lambda(1-p)t}$$

$$\therefore f(t) = \lambda(1-p)e^{-\lambda(1-p)t}, \quad t > 0$$

Thus T has exponential distribution with parameter $\lambda(1-p)$

Put p=0, we get

$$\therefore f(t) = \lambda e^{-\lambda t}$$

Hazard rate of exponential distribution

$$f(t) = \lambda e^{-\lambda t}$$

or,

$$F(t) = 1 - e^{-\lambda t} \ , \ t > 0$$

$$Z(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1 - F(t)} = \lambda$$

This is the phase-II in failure pattern graph

Gamma Distribution

Suppose that the shocks occur in a Poisson Process. And the device fails as soon as the *r*-th shock arrives

$$R(t) = P(T > t)$$

$$\Rightarrow P\left(\text{There will be no failure during the interval (0,t]}\right)$$

$$\Rightarrow P\left(\text{There will be less than } r\text{-shocks in the interval (0,t]}\right)$$

$Let:$

X_t be the random varible "Number of Shocks that arrive in (0,t]"

Then Clearly distribution of X follows poisson with parameter λt

$$R(t) = P(X_t \le r-1)$$

$$= \sum_{k=0}^{r-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

Differentiating with respect to t we get

$$R'(t) = -\frac{\lambda\left(\lambda t\right)^{r-1}}{(r-1)!}e^{-\lambda t},\ t > 0$$

$$f(t) = \frac{\lambda\left(\lambda t\right)^{r-1}}{\Gamma(r)}e^{-\lambda t},\ t > 0$$

Which is a Gamma distribution

Failure rate of Gamma distribution

$$f(t) = \frac{\lambda(\lambda t)^{r-1}}{\Gamma(r)} e^{-\lambda t}, \; t > 0$$

$$Z(t) = \frac{f(t)}{R(t)} = \frac{t^{r-1} e^{-\lambda t}}{\int_t^{\infty} x^{r-1} e^{-\lambda x} dx}$$

$$\text{Let } D = \left(\int_t^{\infty} x^{r-1} e^{-\lambda x} dx \right)^2 .$$

$$Z'(t) = \frac{t^{r-2} e^{-\lambda t} \left[(r-1-\lambda t) \int_t^{\infty} x^{r-1} e^{-\lambda x} dx + t^r e^{-\lambda t} \right]}{D}$$

$$\text{Now let } \psi(\text{t}) = (r-1-\lambda t) \int_t^{\infty} x^{r-1} e^{-\lambda x} dx + t^r e^{-\lambda t}$$

$$\Rightarrow \psi'(\text{t}) = -(r-1) \int_t^{\infty} x^{r-2} e^{-\lambda x} dx, \text{ (using by parts)}$$

Case I: If $0 < r < 1$.

Then $\psi'(t) > 0$ for all $t > 0$.

$\Rightarrow \psi(t)$ is an increasing function and increases from a negative value to zero.

$\Rightarrow \psi(t) < 0$ for all $t > 0$.

$\Rightarrow Z'(t) <$ for all $t > 0$.

$\Rightarrow Z(t)$ is a decresing function of t.

Thus Gamma distribution for $(0 < r < 1)$ can be used as a life- model for any device when the device in phase-I.

Case II: If $r > 1.$

$$let\ \ t < \frac{r-1}{\lambda},\ \ \psi(t) > 0 \Rightarrow Z'(t) > 0\ \ for\, t < \frac{r-1}{\lambda}$$

$let\ \ t \geq \frac{r-1}{\lambda}$, then $\psi'(t) < 0$ as $r > 1$.

$\Rightarrow \psi(t)$ is decreasing function and decreases from a positive value to zero.

$\Rightarrow \psi(t) > 0$ for all $t > 0$.

$\Rightarrow Z'(t) > 0$ for all $t > 0$.

$\Rightarrow Z(t)$ is an incresing function.

Thus Gamma distribution for $(r > 1)$ can be used as a life- model for any device when the device in phase-III.

Case III: If $r = 1.$

Then the Gamma distribution is changed
to exponetial distribution with parameter λ.
$\Rightarrow Z(t) = \lambda$ (constant function).
Thus Gamma distribution for $(r = 1)$
can be used as a life- model for any device
when the device in phase-II.

Failure rate of Normal Distribution

Let T be the life length of any device which follows normal distribution with mean μ and variance σ^2.

$\Rightarrow \dfrac{T-\mu}{\sigma}$ is follows standard normal distribution.

Assume that, the mean is so large.

$\Rightarrow P(T < 0)$ is neglegible, so that $P(T \geq 0) \approx 1.$

$$Z'(t) = \frac{e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}}{\sigma^2\sqrt{2\pi}} \frac{\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} - \left(\frac{t-\mu}{\sigma}\right)\{1-F(t)\}\right]}{(1-F(t))^2}$$

Now let ψ(t)= $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} - \left(\frac{t-\mu}{\sigma}\right)\{1-F(t)\}$

Case I: If $0 < t \leq \mu$.

Then $\dfrac{t-\mu}{\sigma} \leq 0$, and $\psi(t) > 0$.

$\Rightarrow Z'(t) > 0$ for all $0 < t < \mu$.

$\Rightarrow Z(t)$ is an incresing function.

Case II: If $t > \mu$.

Then $\psi'(t) = -\frac{1}{\sigma}(1 - F(t))$.

$\Rightarrow \psi'(t) < 0$ for all $t > \mu$.

$\Rightarrow \psi(t)$ is a decreasing function and decreases from a positive value to zero.

$\Rightarrow \psi(t) > 0$ for all $t > \mu$.

$\Rightarrow Z'(t) > 0$ for all $t > \mu$.

$\Rightarrow Z(t)$ is a increasing function of $t > 0$.

Thus Normal distribution for (μ large) can be used as a life- model for any device when the device in phase-III.

Statistical Methods of Determining Failure Time Distribution

$$\hat{F}(t_i) = \frac{i - 0.5}{n}, \quad \text{using correction for continuity.}$$

$$f_d(t) = \frac{n(t_i) - n\left(t + \Delta t_i\right)}{n \Delta t_i}, \; t_i \le t \le t_i + \Delta t_i$$

$$Z_d(t) = \frac{n(t_i) - n\left(t + \Delta t_i\right)}{n(t_i) \Delta t_i}, t_i \le t \le t_i + \Delta t_i$$

Data Analysis to check suitability of exponential distribution

Ex. V11:

100 components were put on life-test. The test was terminated as soon as the 11th components failed. The life-length of those components which failed were as follows:

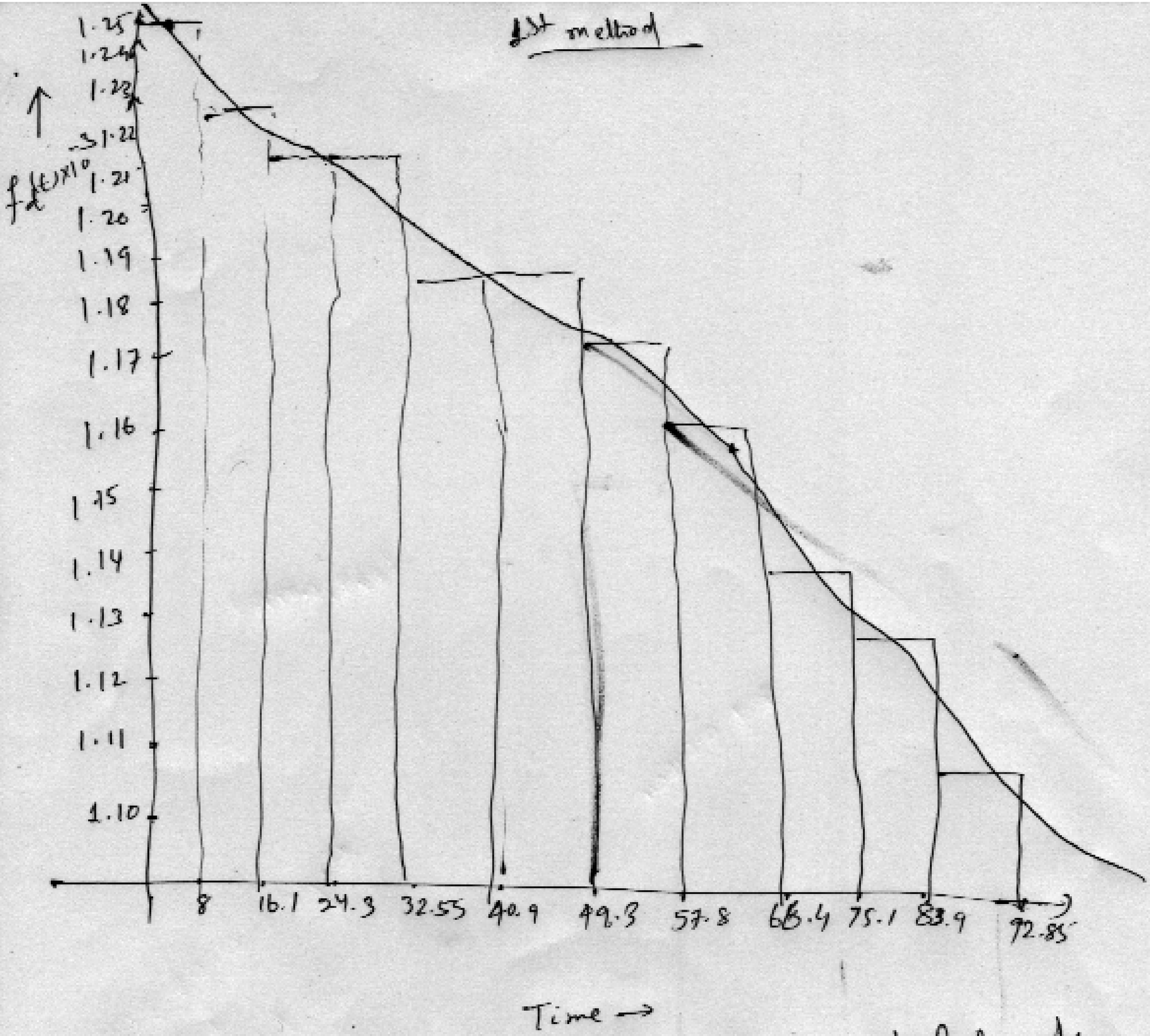
8, 16.1, 24.3, 32.55, 40.9, 57.8, 66.4, 75.1, 83.9, 92.85. Guess which distribution fits the data. Estimate the parameters.

Failure No.	1	2	3	4	5	6
Time to failure	8	16.1	24.3	32.55	40.9	49.3
Failure No.	7	8	9	10	11	
Time to Failure	57.8	66.4	75.1	83.9	92.85	

Case-I:

Time interval	$f_d(t)$	Time interval	$f_d(t)$
0-8	0.00125	49.3-57.8	0.00117
8-16.1	0.00123	57.8-66.4	0.00116
16.1-24.3	0.00121	66.4-75.1	0.00114
24.3-32.55	0.00121	75.1-83.9	0.00113
32.55-40.9	0.00119	83.9-92.85	0.00111
40.9-49.3	0.00119		

1st method



Since the graph looks like exponential density

⇒ exponential distribution fits the data

& roughly it meets y-axis at the point

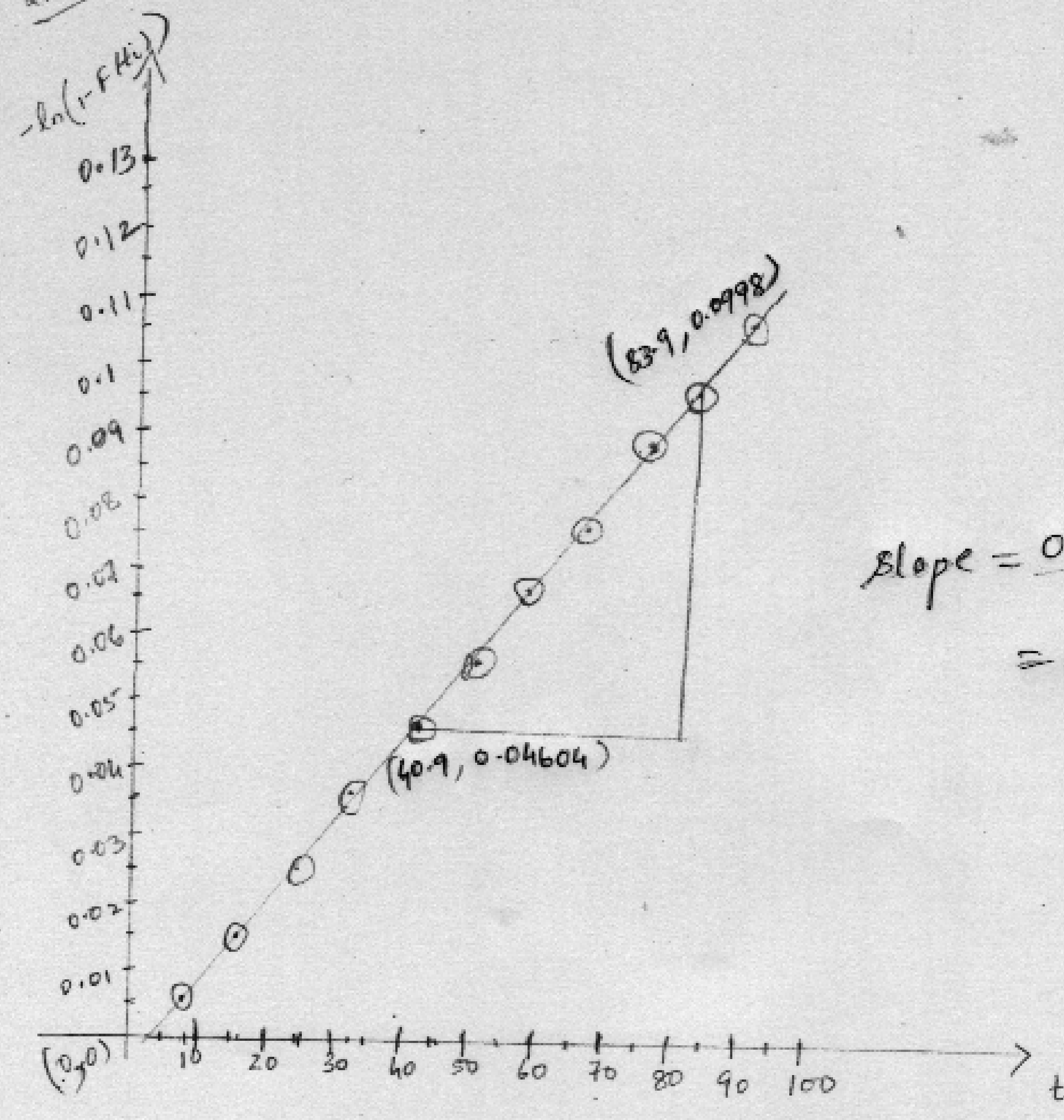
$\approx 1.27 \times 10^{-3}$

∴ we can assume $\lambda \approx 1.27 \times 10^{-3}$

Case-II:

t_i	$-\ln(1-\hat{F}(t_i))$	t_i	$-\ln(1-\hat{F}(t_i))$
8	0.00501	57.8	0.0672
16.1	0.0151	66.4	0.0780
24.3	0.0253	75.1	0.0888
32.55	0.0356	83.9	0.0998
40.9	0.0460	92.85	0.1109
49.3	0.0566		

2nd Method.



$$slope = \frac{0.05376}{43} = 1.25 \times 10^{-3}$$

$$1 - F(t) = e^{-\lambda t}$$

$\ln(1-F(t)) = -\lambda t$ or $-\ln(1-F(t)) = \lambda t$

plote $(t_i, -\ln(1-F(t_i)))$ → should be a stgt line

& slope of that line $= \lambda = 1.25 \times 10^{-3}$

Case-III:

Time interval	$Z_d(t)$	Time interval	$Z_d(t)$
0-8	0.00125	49.3-57.8	0.00125
8-16.1	0.00125	57.8-66.4	0.00125
16.1-24.3	0.00124	66.4-75.1	0.00125
24.3-32.55	0.00126	75.1-83.9	0.00125
32.55-40.9	0.00124	83.9-92.85	0.00124
40.9-49.3	0.00125		

Since $Z_d(t)$ is almost constant, we can assume exponential distribution fits the data. Now using least square method we can determine the parameter λ.

Calculation for λ .

$$\hat{\lambda} = \frac{\sum_{i=1}^{11} Z(t_i)}{11} = 0.00125.$$

Data Analysis to check suitability of weibull distribution with α=0.

Weibull distribution with parameter (α, β, η).

A random variable T is said to have Weibull distribution if its density function is given by

$$f(t) = \begin{cases} \dfrac{\beta}{\eta}\left(\dfrac{t-\alpha}{\eta}\right)^{\beta-1} e^{-\left(\frac{t-\alpha}{\eta}\right)^{\beta}}, t > \alpha \\ 0, t \le \alpha \end{cases}$$

Special case of Weibull distribution when $\alpha = 0$.

$$f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta - 1} e^{-\left(\frac{t}{\eta}\right)^{\beta}}, t > 0.$$

It can be proved that the hazard rate and reliability of the above Weibull distribution are

$$Z(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta - 1}, R(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}}, t > 0.$$

Consider the reliability $R(t)$.

$$R(t) = e^{-\left(\frac{t}{\eta}\right)^{\beta}}, t > 0.$$

$$\ln(R(t)) = -\left(\frac{t}{\eta}\right)^{\beta} \Rightarrow -\ln(R(t)) = \left(\frac{t}{\eta}\right)^{\beta}$$

$$\Rightarrow \ln\left(\ln\left(\frac{1}{R(t)}\right)\right) = \beta\ln(t) - \beta\ln\eta$$

$$\Rightarrow \ln\left(\ln\left(\frac{1}{1-F(t)}\right)\right) = \beta \ln(t) - \beta \ln \eta$$

$$\Rightarrow \ln\left(\ln\left(\frac{1}{1-\hat{F}(t_i)}\right)\right) = \beta \ln(t_i) - \beta \ln \eta$$

Now plot $\left(\ln(t_i), \ln\left(\ln\left(\frac{1}{1-\hat{F}(t_i)}\right)\right)\right)$

If the points $\left(\ln(t_i), \ln\left(\ln\left(\dfrac{1}{1 - \hat{F}(t_i)} \right) \right) \right)$

lies almost on a straight line, we can assume Weibull distribution with α=0 fits the data.

<u>How to calculate the parameter β and η.</u>

Case-I (using graphical method)

$\hat{\beta} =$ slope of the above line

and $\hat{\eta}$ can be obtained by putting any value for t_i

in the equation given below:

$$\hat{\eta} = \ln(t_i) - \frac{1}{\hat{\beta}} \ln\left(\ln\left(\frac{1}{1 - \hat{F}(t_i)} \right)\right)$$

Case-ii (using least square method)

Least square method

Suppose $(x_i, y_i), i = 1, 2, \cdots, n$ given. We want to fit one straight line $bx + a$ to the above data. Least square method estimates a and b by solving

$$\min_{(a,b)} \sum_{i=1}^{n} \left(y_i - bx_i - a \right)^2 . \qquad (1)$$

Using derivative concept, we can easily say that equation (1) gives minimum when

$$\hat{b} = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$\hat{a} = \frac{\sum y_i - \hat{b}\sum x_i}{n}$$

In our problem,

$$x_i = \ln(t_i)$$

$$y_i = \ln\left(\ln\left(\frac{1}{1-\hat{F}(t_i)} \right) \right)$$

$$b = \beta$$

$$a = -\beta \ln(\eta)$$

Using least square method, we can estimate

$$\hat{\beta} = \hat{b}$$

$$\hat{\eta} = e^{-\frac{\hat{a}}{\hat{\beta}}}$$

Consider the reliability $Z(t)$.

$$Z(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}, t > 0.$$

$$\ln(Z(t)) = \ln\left(\frac{\beta}{\eta}\right) + (\beta-1)\ln(t) - (\beta-1)\ln\eta$$

$$\Rightarrow \ln(Z(t)) = (\beta-1)\ln(t) + \ln\beta - \beta\ln\eta$$

$$\ln(Z_d(t_i)) = (\beta - 1)\ln(t_i) + \ln\beta - \beta\ln\eta$$

Now plot $\left(\ln(t_i), \ln\left(Z_d(t_i)\right)\right)$

If the points $\left(\ln(t_i), \ln\left(Z_d(t_i)\right)\right)$

lies almost on a straight line, we can assume Weibull distribution with α=0 fits the data.

<u>How to calculate the parameter β and η.</u>

Case-I (using graphical method)

$\hat{\beta} - 1 =$ slope of the above line

$\hat{\beta} =$ 1+slope of the above line and $\hat{\eta}$ can be obtained by putting any value for t_i

in the equation given below:

$$\ln\hat{\eta} = \frac{\ln\hat{\beta} + (\hat{\beta} - 1)\ln(t_i) - \ln\left(Z_d(t_i)\right)}{\hat{\beta}}$$

Case-ii (using least square method)

Least square method

Suppose $(x_i, y_i), i = 1, 2, \cdots, n$ given. We want to fit one straight line $bx + a$ to the above data. Least square method estimates a and b by solving

$$\min_{(a,b)} \sum_{i=1}^{n} \left(y_i - bx_i - a \right)^2 . \qquad (1)$$

Using derivative concept, we can easily say that equation (1) gives minimum when

$$\hat{b} = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^{\,2} - \left(\sum x_i\right)^2}$$

$$\hat{a} = \frac{\sum y_i - \hat{b}\sum x_i}{n}$$

Using least square method, we can estimate

$$\hat{\beta} = \hat{b} + 1$$

$$\hat{\eta} = e^{\frac{\ln \hat{\beta} - \hat{a}}{\hat{\beta}}}$$

In our problem,

$$x_i = \ln(t_i)$$

$$y_i = \ln\left(Z_d(t_i)\right)$$

$$b = \beta - 1$$

$$a = \ln \beta - \beta \ln \eta$$

Ex. V12:

100 components were put on life-test. The test was terminated as soon as the 7th components failed. The life-length of those components which failed were 2, 3.3, 4.7, 5.9, 7, 8, 9.

(a) Using hazard rate, check whether Weibull distribution fits the data or not ? Estimate the parameters using

(i) graphical method.

(ii) least square method.

(b) Using cdf/reliability, check whether Weibull distribution fits the data or not ? Estimate the parameters using

(i) graphical method.

(ii) least square method.

Failure No.	1	2	3	4	5	6	7
Time to failure	2	3.3	4.7	5.9	7	8	9

Time interval	$Z_d(t)$
0 – 2	0.005
2 - 3.3	0.0077
3.3 – 4.7	0.0072
4.7 – 5.9	0.0085
5.9 – 7	0.0094
7 - 8	0.0105
8 - 9	0.0106

Solution (a):

t_i	$\ln(t_i)$	$\ln(Z_d(t_i))$
2	0.69	-5.29
3.3	1.19	-4.85
4.7	1.54	-4.87
5.9	1.77	-4.76
7	1.94	-4.66
8	2.07	-4.55
9	2.19	-4.54

since the points $\left(\ln(t_i), \ln\left(Z_d(t_i)\right)\right)$ lies almost on a straight line, we can assume Weibull distribution fits the data.

<u>a(i). Calculation for parameters.</u>

$$\hat{\beta} = 1 + slope = 1 + 0.01 = 1.01$$

$$\hat{\eta} = e^{\frac{\ln \hat{\beta} + (\hat{\beta} - 1)\ln(t_i) - \ln\left(Z_d(t_i)\right)}{\hat{\beta}}}$$

$$= e^{\frac{\ln 1.01 - (0.01)\times 0.69 + 5.29}{1.01}} = 188.79$$

a(ii). Calculation for parameters

$$x_i = \ln(t_i),\ y_i = \ln\left(Z_d(t_i)\right),\ i = 1, 2, \cdots, 7.$$

$$\sum_{i=1}^{7} x_i = 11.39,\ \sum_{i=1}^{7} y_i = -33.52$$

$$\sum_{i=1}^{7} x_i^{\,2} = 20.24,\ \sum_{i=1}^{7} x_i y_i = -53.74$$

$$\therefore\ \hat{b} = 0.44,\ \hat{a} = -7.13$$

$$\Rightarrow\ \hat{\beta} = 1.44,\ \hat{\eta} = e^{\frac{\ln(1.44)+7.13}{1.44}} = 182.32.$$