MECHANICS OF SOLIDS (ME F211)

BITS Pilani
K K Birla Goa Campus

VIKAS CHAUDHARI

Chapter-8

Deflections due to Bending

Deflections due to Bending

Contents:

- The moment curvature relation
- Integration of moment curvature relation
- Principle of Superposition
- Load- deflection differential equation
- Energy methods

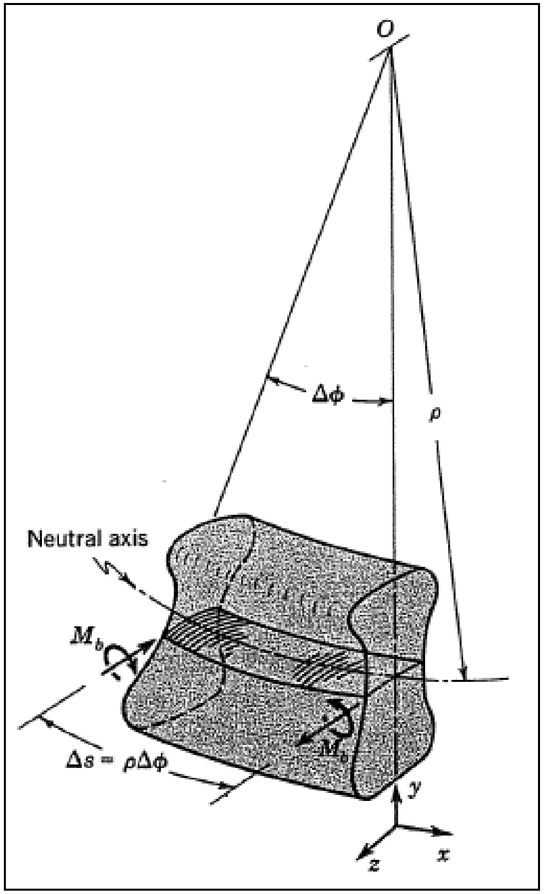
Deflections due to Bending

The moment curvature relation

The relation between curvature of neutral axis and bending moment is given by

$$\frac{1}{\rho} = \lim_{\Delta s \to 0} \frac{\Delta \phi}{\Delta s} = \frac{d\phi}{ds} = \frac{M_b}{EI_{zz}}$$

- *E* is modulus of elasticity and I_{zz} is moment of inertia.
- Longitudinal dimension of the beam will be in *x* direction.
- Bending will take place in *xy* plane about *z* axis.
- Instead of symbol I_{zz} for moment of inertia, we use the abbreviation *I*.



Deformation of an element of a beam subjected to bending moments M_b

Deflections due to Bending

The moment curvature relation

Deflection of neutral axis from the knowledge of its curvature

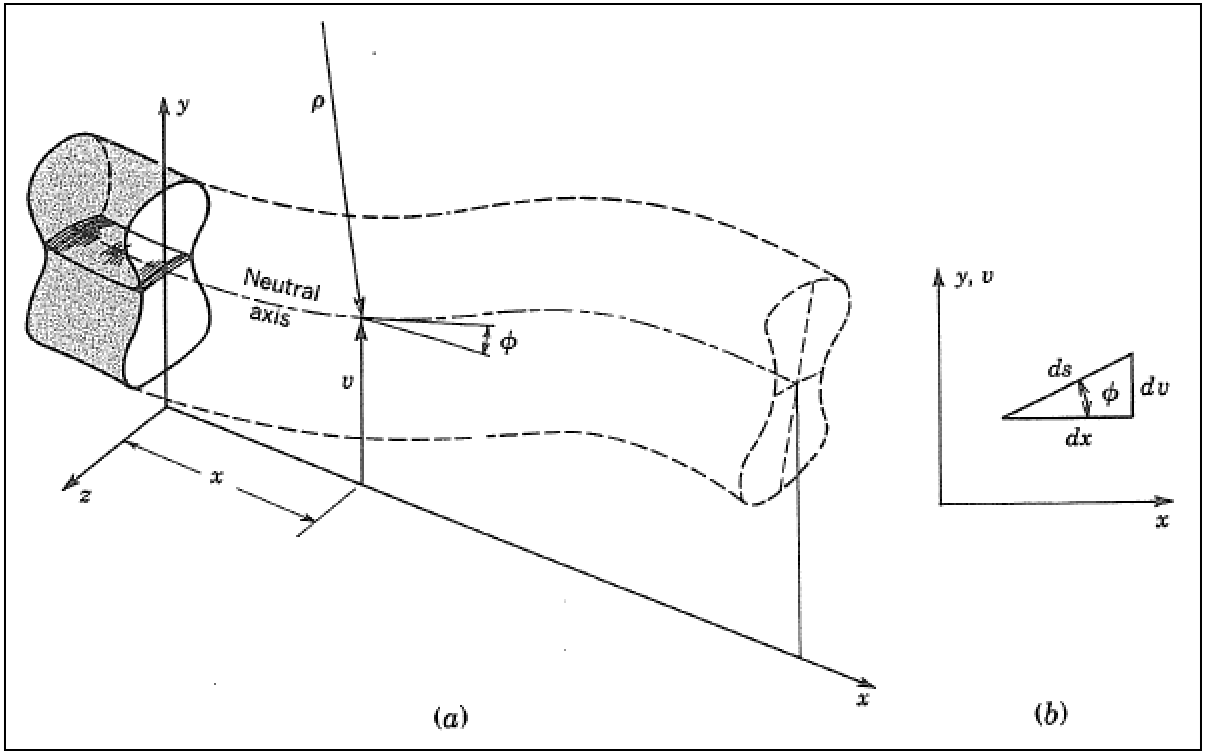
Slope of the neutral axis

$$\frac{dv}{dx} = \tan\phi$$

Differential with arc length *s*.

$$\frac{d^2v}{dx^2}\frac{dx}{ds} = \sec^2\phi\frac{d\phi}{ds}$$

$$\frac{d\phi}{ds} = \frac{d^2v}{dx^2}\frac{dx}{ds}\cos^2\phi$$



Geometry of the neutral axis of a beam bent in the *xy* plane

Deflections due to Bending

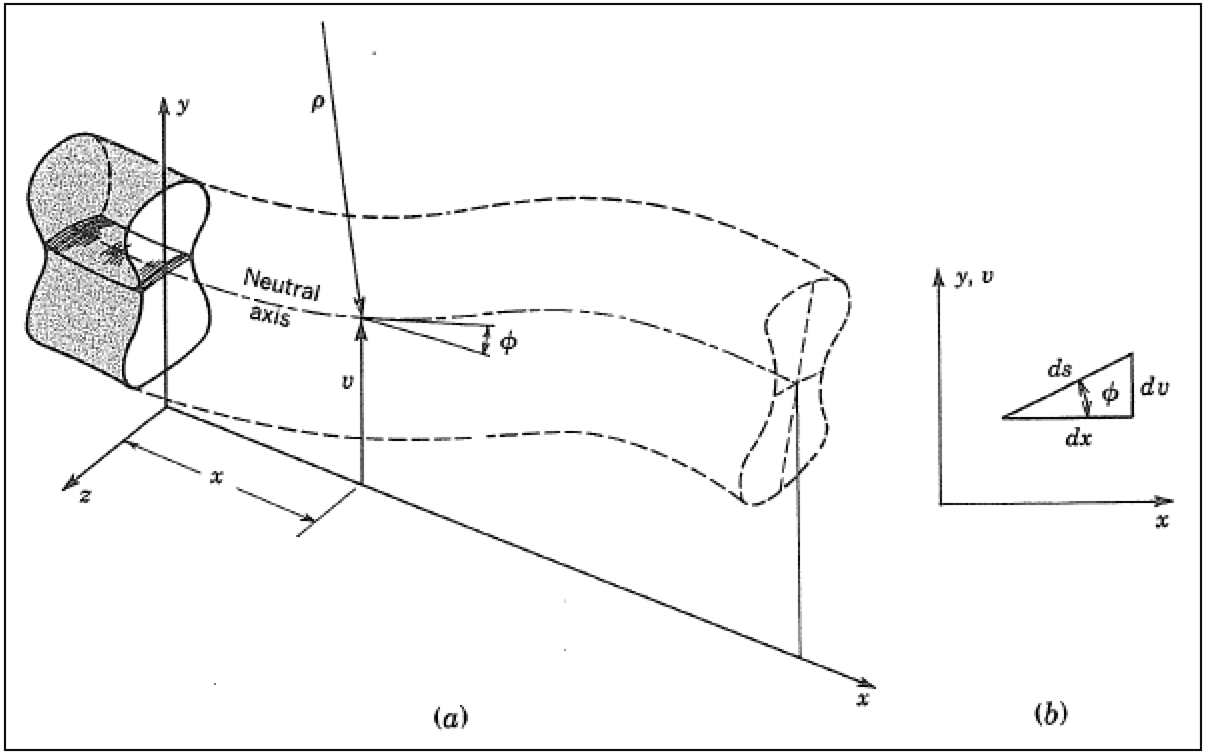
The moment curvature relation

From Figure

$$\cos\phi = \frac{dx}{ds} = \frac{1}{\left[1+\left(dv/dx\right)^2\right]^{1/2}}$$

The curvature is

$$\frac{d\phi}{ds} = \frac{d^2v/dx^2}{\left[1+\left(dv/dx\right)^2\right]^{3/2}}$$



Geometry of the neutral axis of a beam bent in the *xy* plane

Deflections due to Bending

The moment curvature relation

$$\frac{d\phi}{ds} = \frac{M_b}{EI_{zz}} \quad \Rightarrow \quad \frac{d^2v/dx^2}{\left[1+\left(dv/dx\right)^2\right]^{3/2}} = \frac{M_b}{EI}$$

- Above equation is nonlinear differential equation for determination of v as a function of x^2, if M_b is known.
- When the slope angle ϕ is small, then dv/dx is small compared to unity

$$\frac{d\phi}{ds} \approx \frac{d^2v}{dx^2}$$

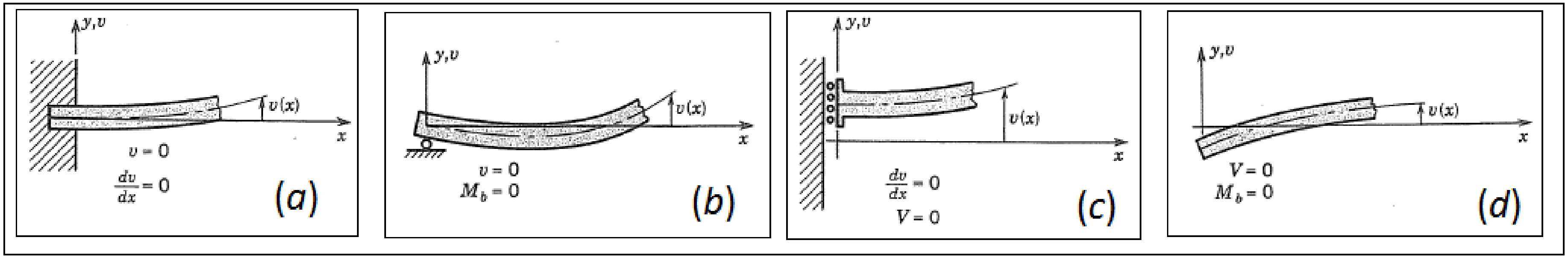
$$\frac{d^2v}{dx^2} = \frac{M_b}{EI} \quad \text{-------} \quad 1$$

- Equation 1 is called moment curvature relation
- The term *EI* is referred as *flexural rigidity* or *bending modulus.*

Deflections due to Bending

Integration of the moment curvature relation

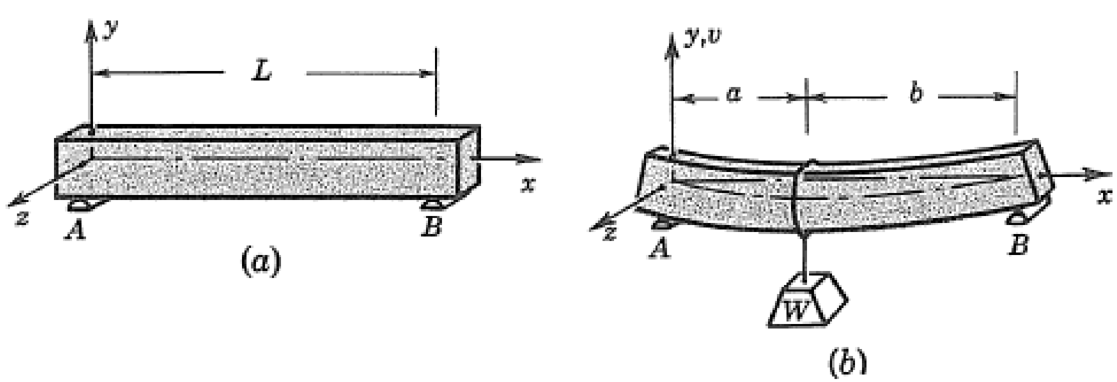
- Integration of moment curvature relation leads to the correct deflection curve.
- However suitable boundary conditions should be chosen to determine the integration constants.
- Figure shows the suitable boundary condition encountered in various supports.



Deflections due to Bending

Problem:

The simply supported beam of uniform cross section shown in figure is subjected to a concentrated load *W*. It is desired to obtained maximum slope and maximum deflection.



Simply supported beam (*a*) before and (*b*) after application of load *W*

$L = 3.70\text{m}$
$a = b = 1.85\text{m}$
$W = 1.8\text{kN}$
$E = 11\text{GPa}$
$I = 3.33 \times 10^7 \text{mm}^4$

Deflections due to Bending

Solution:

- Singularity function method is used to find bending moment.
- The maximum deflection is the point at which the slope is zero.
- Therefore maximum deflection will be at $x = L/2$.
- Maximum slope will be at $x = 0$ or at $x = L$.

Maximum Deflection v_{max} will be

$$v_{\max} = (v)_{x=L/2} = -\frac{WL^3}{48EI} = 5.19mm$$

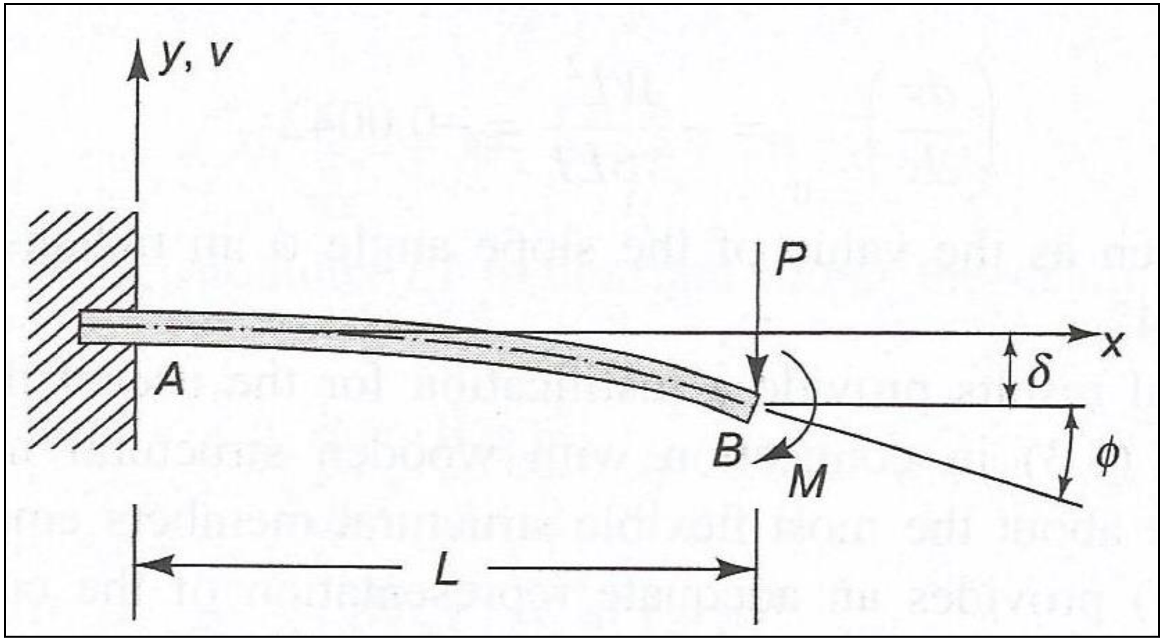
Maximum slope ϕ_{max} will be

$$\phi_{\max} = \left(\frac{dv}{dx}\right)_{x=0} = -\frac{WL^2}{16EI} = -0.0042rad = 0.24^o$$

Deflections due to Bending

Problem:

A uniform cantilever beam has bending modulus *EI* and length *L*. It is built in at *A* and subjected to a concentrated force *P* and moment *M* applied at B as shown in figure. Find deflection δ and slope ϕ at point *B*.



Deflections due to Bending

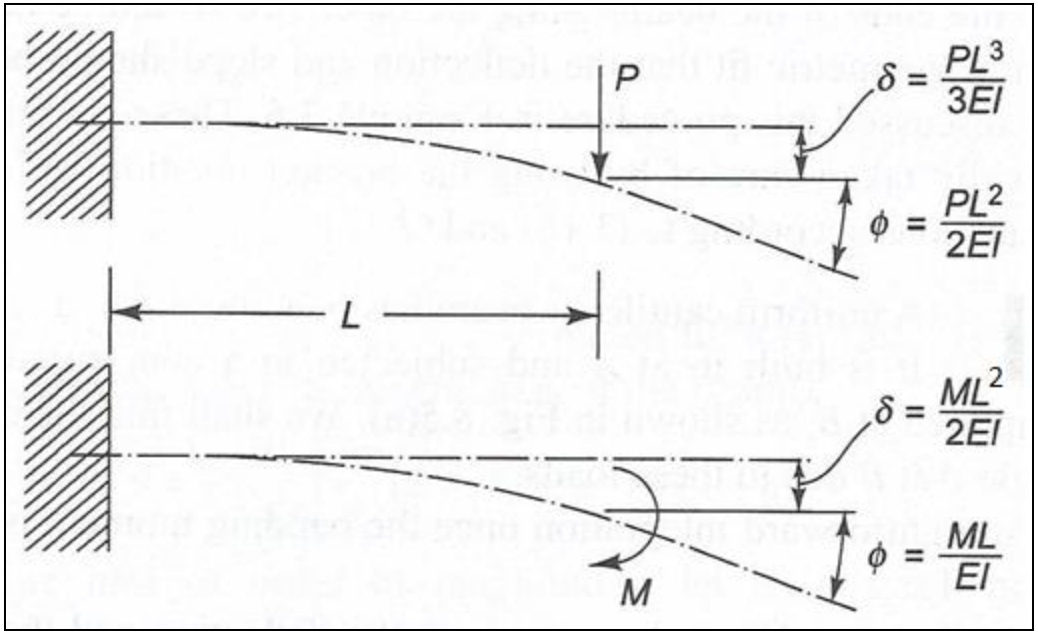
Solution:

- Deflection δ at point B will be

$$\delta_B = -(v)_{x=L} = \frac{PL^3}{3EI} + \frac{ML^2}{2EI}$$

- Deflection ϕ at point B will be

$$\phi_B = -\left(\frac{dv}{dx}\right)_{x=L} = \frac{PL^2}{2EI} + \frac{ML}{EI}$$



Deflections due to Bending

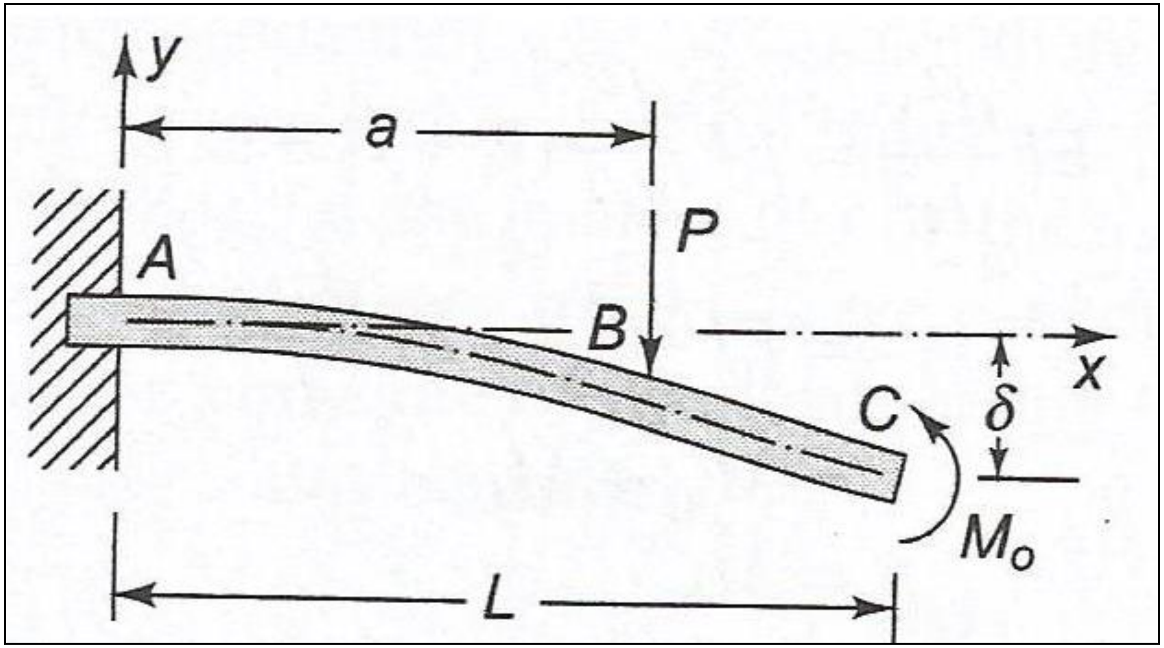
Superposition

- The total deflection is the sum of deflections due to individual load (M_b)
- The deflections in the standard cases are given in table 8.1. The solution of the original problem then takes the form of a superposition of these solutions.
- Deflection of a beam is linearly proportional to the applied load
- The linearity between curvature and deflection is based on assumption that
  - Deflections are small
  - material is linearly elastic

Deflections due to Bending

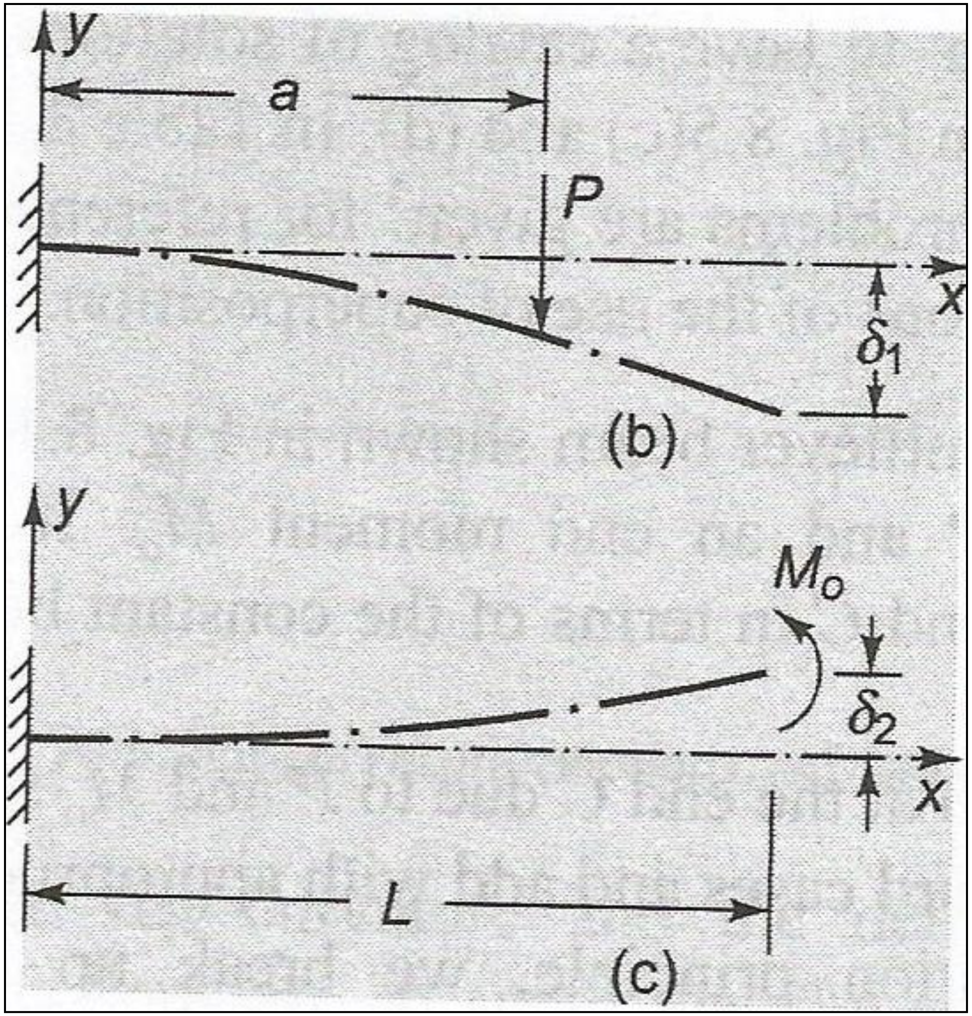
Problem:

The cantilever beam shown in figure carries a concentrated load P and end moment M_o applied at B as shown in figure. Find deflection δ at point C in terms of the constant bending modulus EI.



Deflections due to Bending

Solution:



The Load-Deflection Differential Equation

- An alternative method to solve beam deflection problem.
- The differential equations for force and moment equilibrium are

$$\frac{dV}{dx} + q = 0 \quad \text{and} \quad \frac{dM_b}{dx} + V = 0$$

Therefore,

$$\frac{d^2 M_b}{dx^2} = q$$

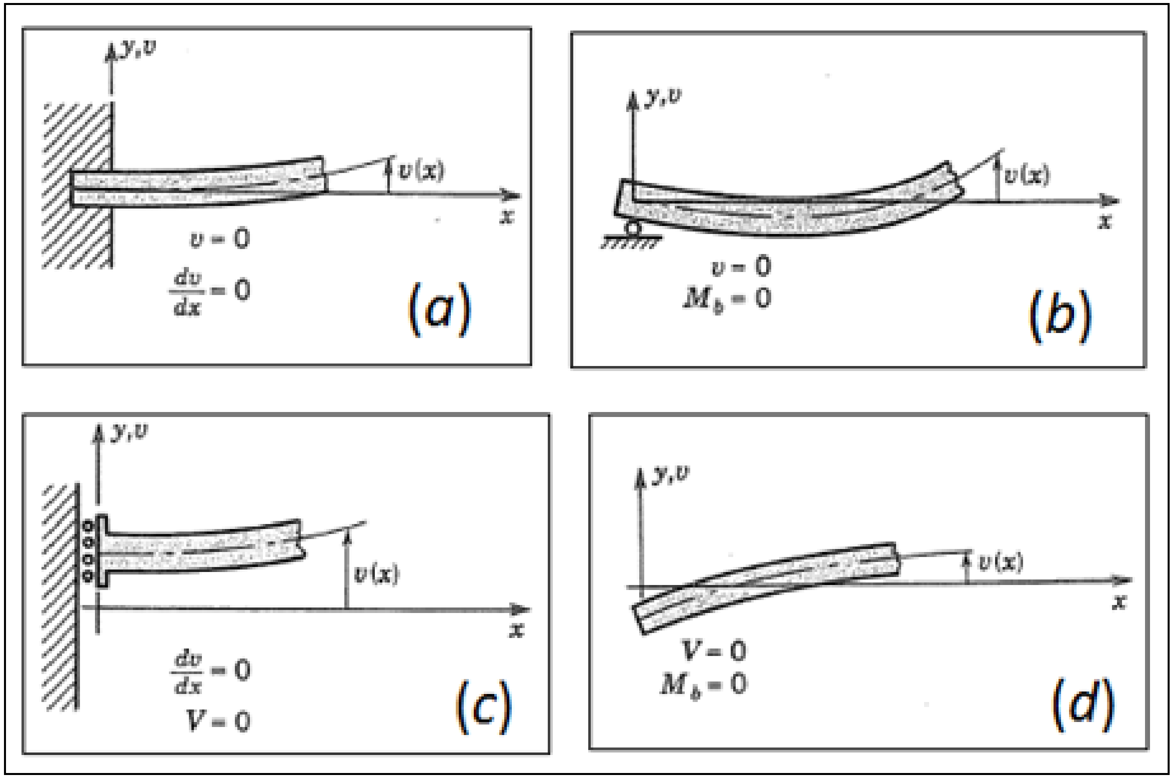
- Using above equation and moment curvature relation, we obtained a single differential equation relating transvers load-intensity function *q* and transverse deflection *v*.

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2 v}{dx^2}\right) = q$$

Deflections due to Bending

Boundary Conditions

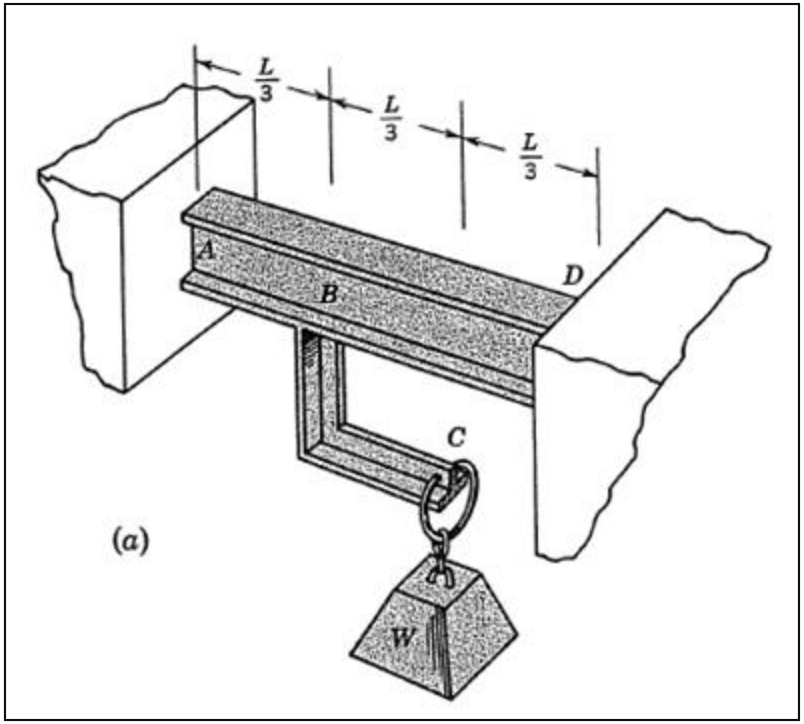
- Figure shows the suitable boundary conditions corresponds to four types of supports.



Deflections due to Bending

Problem

The beam shown in figure is built-in at *A* and *D* and has an offset arm welded to the beam at the point *B* with a load *W* attached to the arm at *C*. It is required to find the deflection of the beam at the point *B*.



Deflections due to Bending

Solution

Load intensity

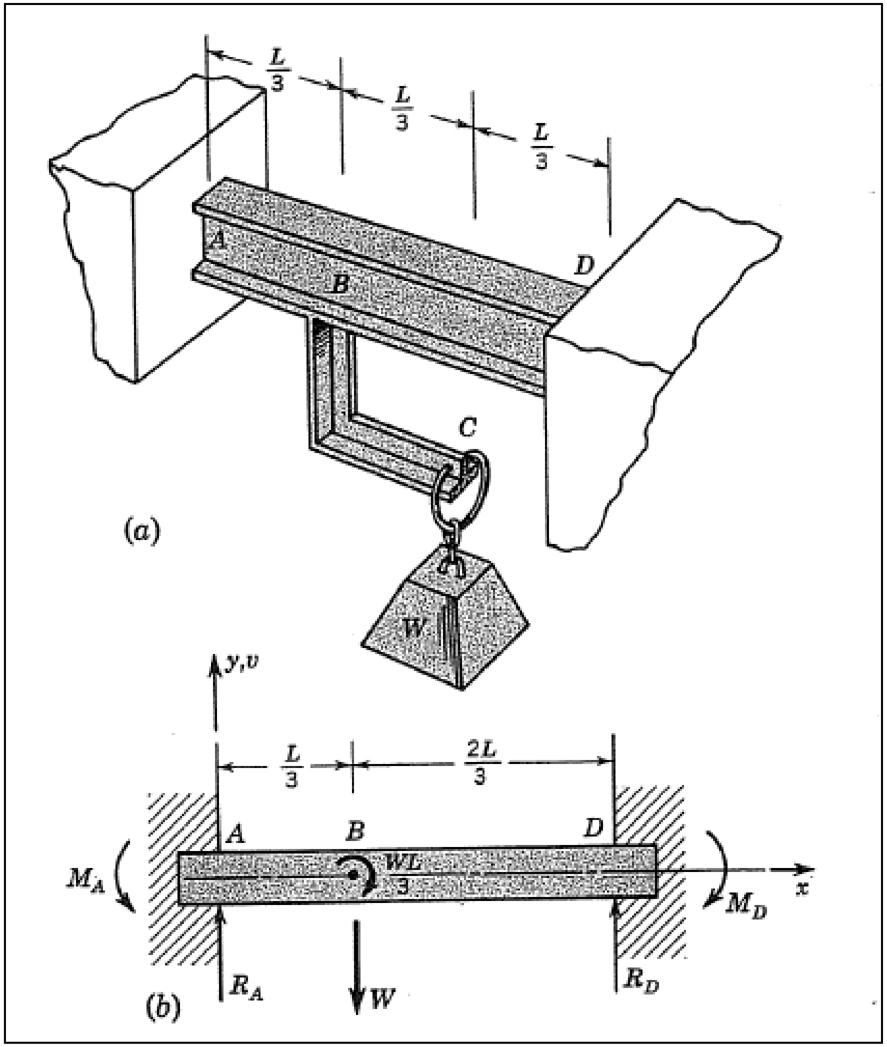
$$q = \frac{WL}{3}\langle x - L/3 \rangle_{-2} - W\langle x - L/3 \rangle_{-1}$$

Boundary conditions

$$v = 0 \quad \text{and} \quad \frac{dv}{dx} = 0 \qquad \text{at} \;\; x = 0 \;\; \text{and} \;\; L$$

Load-deflection differential equation

$$EI\frac{d^4 v}{dx^4} = W\left(\frac{L}{3}\langle x - L/3 \rangle_{-2} - \langle x - L/3 \rangle_{-1}\right)$$



Deflections due to Bending

Solution

By integrating previous equation

$$\frac{dv}{dx}=\frac{W}{EI}\left(\frac{L}{3}\langle x-L/3\rangle^{1}-\frac{\langle x-L/3\rangle^{2}}{2}+c_1\frac{x^2}{2}+c_2x+c_3\right)$$

$$v=\frac{W}{EI}\left(\frac{L}{6}\langle x-L/3\rangle^{2}-\frac{\langle x-L/3\rangle^{3}}{6}+c_1\frac{x^3}{6}+c_2\frac{x^2}{2}+c_3x+c_4\right)$$

Boundary conditions gives four integration constants

$$c_1=\frac{8}{27};\qquad c_2=-\frac{4}{27}L;\qquad c_3=c_4=0$$

Deflections due to Bending

Solution

By inserting boundary conditions in deflection equation

$$v = \frac{W}{27EI}\left(\frac{9}{2}L\left\langle x - L/3\right\rangle^{2} - \frac{9}{2}\left\langle x - L/3\right\rangle^{3} + \frac{4}{3}x^{3} - 2Lx^{2}\right)$$

Deflection at point *B*, by setting $x = L/3$

$$\delta_B = -\left(v\right)_{x=L/3} = \frac{14WL^3}{2187EI}$$

Deflections due to Bending

Castigliano's Method to find the deflections

- Strain energy due to transverse loads

$$U = \frac{1}{2}\iiint \sigma_x \varepsilon_x dxdydz = \iiint \frac{\sigma_x^{\ 2}}{2E} dxdydz$$

$$U = \iiint \frac{1}{2E}\left(\frac{M_b y}{I}\right)^2 dxdydz = \int_L \frac{M_b^{\ 2}}{2EI^2} dx \iint_A y^2 dydz$$

$$U = \int_L \frac{M_b^{\ 2}}{2EI} dx$$

- If total elastic energy in a system is expressed in terms of external loads P_i, the corresponding in-line deflections δ_i are given by partial derivatives

$$\delta_i = \frac{\partial U}{\partial P_i}$$

Castigliano's Method to find the deflections

Important

- If, we may wish to know deflection at a point where external force is zero .
- In such case a fictitious force *Q* is to be considered at that point.
- Deflection at that point in the direction of *Q* is given by ∂U/∂*Q* and setting *Q* = 0.

Similarly slope is given by

$$\phi = \frac{\partial U}{\partial M}$$

Deflections due to Bending

Castigliano's Method to find the deflections

Simplified equation

- Deflection

$$\delta_i = \frac{\partial U}{\partial P_i} = \int_0^L \frac{2M_b}{2EI}\frac{\partial M_b}{\partial P_i}dx = \int_0^L \frac{M_b}{EI}\frac{\partial M_b}{\partial P_i}dx$$

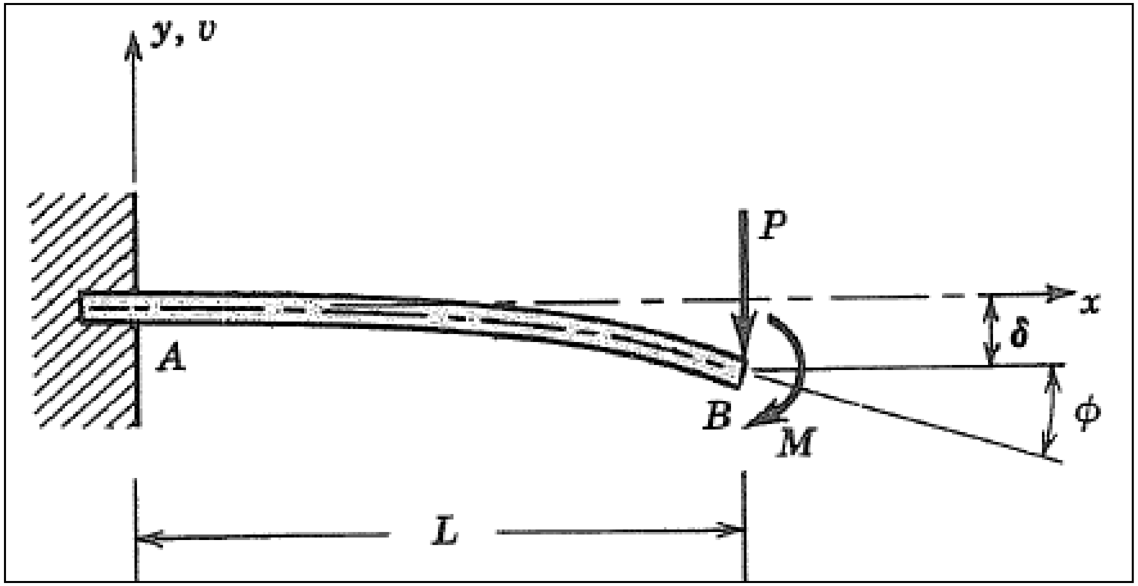
- Slope

$$\phi_i = \frac{\partial U}{\partial M_i} = \int_0^L \frac{2M_b}{2EI}\frac{\partial M_b}{\partial M_i}dx = \int_0^L \frac{M_b}{EI}\frac{\partial M_b}{\partial M_i}dx$$

Deflections due to Bending

Problem

Using Castigliano's method, determine the slope and deflection at point *B* (loading diagram is shown in figure below).



Deflections due to Bending

Solution

Bending moment = $M_b = -P(L-x) - M$ and $\frac{\partial M_b}{\partial P} = -(L-x)$

Deflection at point *B* will be

$$\delta_B = \frac{\partial U}{\partial P} = \int_0^L \frac{2M_b}{2EI}\frac{\partial M_b}{\partial P_i}dx = \int_0^L \frac{M_b}{EI}\frac{\partial M_b}{\partial P}dx$$

After solving above equation

$$\delta_B = \frac{PL^3}{3EI} + \frac{ML^2}{2EI}$$

Deflections due to Bending

Solution

Bending moment = $M_b = -P(L-x) - M$ and $\dfrac{\partial M_b}{\partial M} = -1$

Slope at point *B* will be

$$\phi_B = \frac{\partial U}{\partial M} = \int_0^L \frac{2M_b}{2EI}\frac{\partial M_b}{\partial M}dx = \int_0^L \frac{M_b}{EI}\frac{\partial M_b}{\partial M}dx$$

After solving above equation

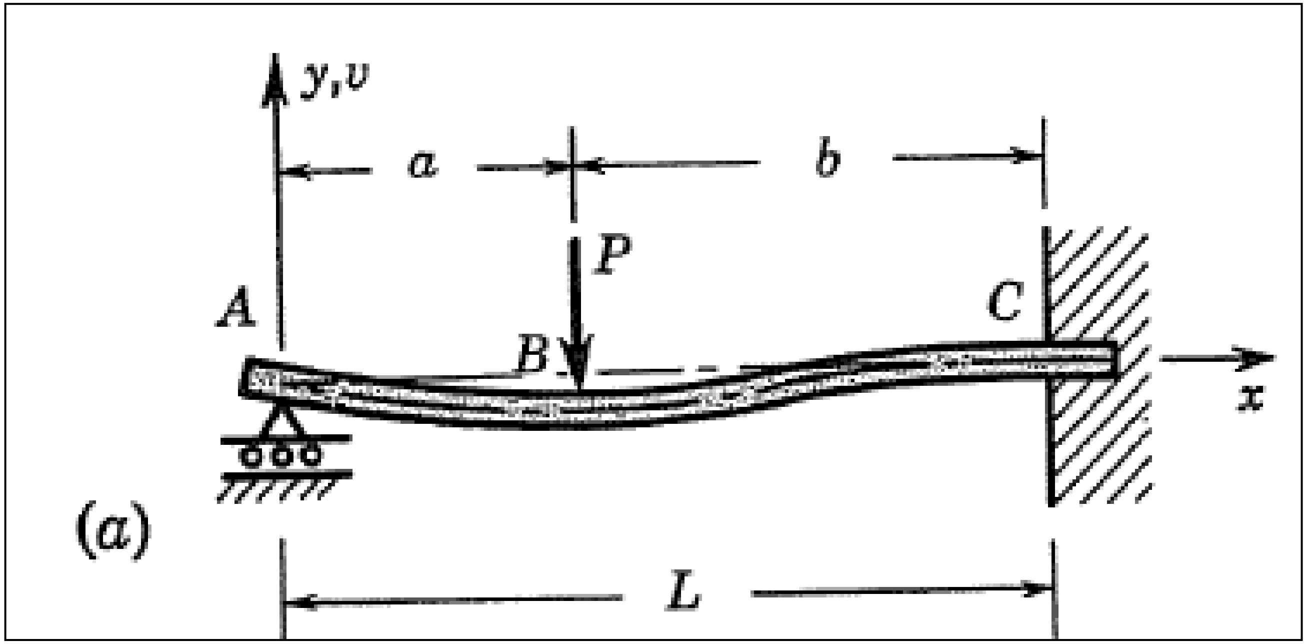
$$\phi_B = \frac{PL^2}{2EI} + \frac{ML}{EI}$$

Deflections due to Bending

Problem

Using Castigliano's method, determine the reaction at point *A*.

Take $a = b = L/2$



Deflections due to Bending

Solution

We know that deflection at point A is zero. Let's find reaction at *A*.

Bending moment = $M_b = R_A x - P\langle(x-a)\rangle^1$ and $\dfrac{\partial M_b}{\partial R_A} = x$

Deflection at point *A* will be

$$\delta_A = \int_0^L \frac{M_b}{EI}\frac{\partial M_b}{R_A}dx = 0$$

$$\delta_A = 0 = \frac{1}{EI}\int_0^L (R_A x^2 - Px < x - \frac{L}{2} >)dx$$

Deflections due to Bending

Solution

$$0 = R_A\left[\frac{x^3}{3}\right]_0^L - P\left[x\int < x-\frac{L}{2} > -\int \frac{dx}{dx}\int < x-\frac{L}{2} >\right]_0^L$$

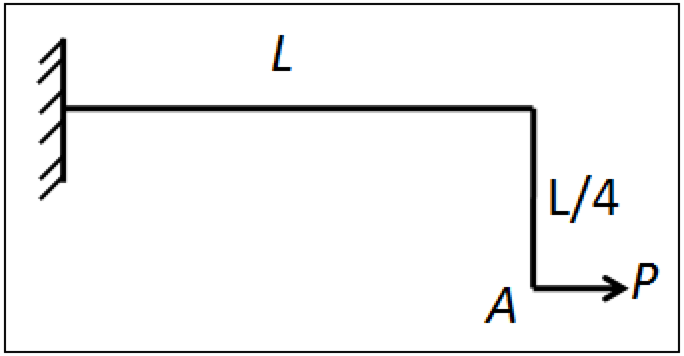
$$0 = \frac{R_A L^3}{3} - \frac{5PL^3}{48}$$

$$R_A = \frac{5}{16}P$$

Deflections due to Bending

Problem

Using Castigliano's method, determine horizontal deflection at point *A* (consider deflection due to only bending moments).



ANS. $$\delta_A = \frac{13PL^3}{192EI}$$

References

1. Introduction to Mechanics of Solids by S. H. Crandall et al (In SI units), McGraw-Hill