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Continuous Random Variables

Definition:

A random variable is continuous if it can assume any value in interval of real numbers.

Or

Range of X is an interval or union of intervals.

Continuous Random Variables

Definition:

Let *X* be a continuous random variable. A function *f* such that

$$1.\ f(x) \geq 0 \text{ for all } x$$

$$2.\ \int_{-\infty}^{\infty} f(x)dx = 1$$

$$3.\ P[a \leq X \leq b] = \int_a^b f(x)dx$$

is called a density for *X*.

Note: $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$.

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$$P(a \le X \le b) = \text{area under } f(x) \text{ from } a \text{ to } b$$

$$= \int_a^b f(x)dx$$

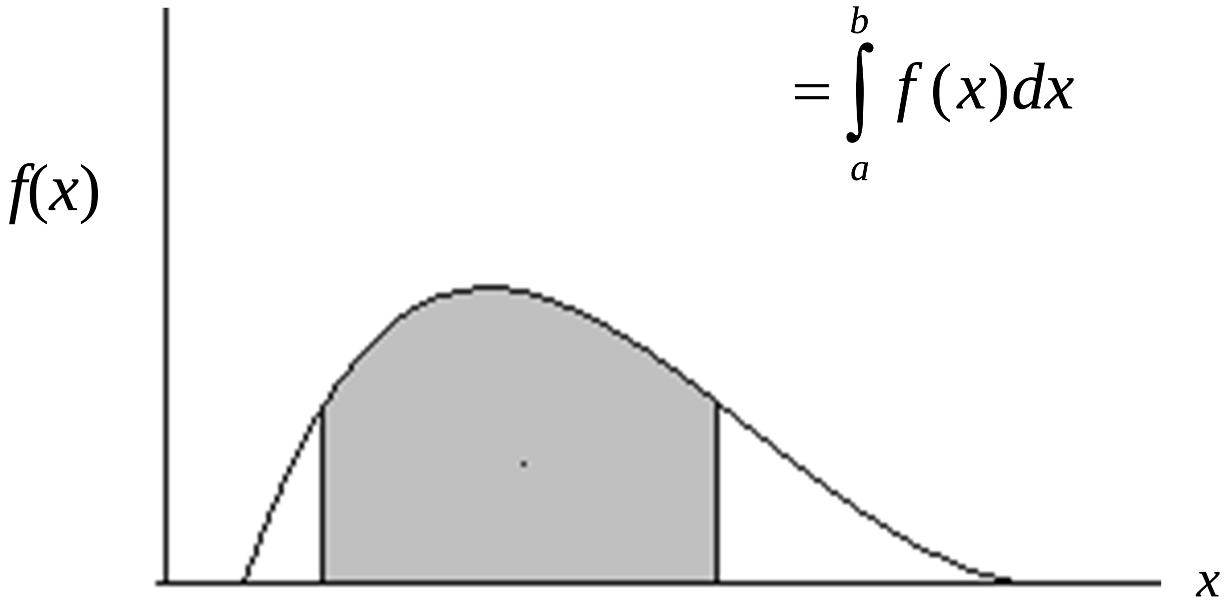

Figure: Probability as area under *f*

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Necessary and sufficient conditions for function to be a continuous density:

$$1.\ f(x) \geq 0$$

$$2.\ \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Cumulative distribution function

Thus, for any value x,

$$F(x) = P(X \leq x)$$

is the area under the probability density function over the interval $-\infty$ to x. Mathematically,

$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

- The probability that the random variable will take on a value on the interval from a to b is given by

$$P(a \leq X \leq b) = F(b) - F(a)$$

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- According to the fundamental theorem of integral calculus it follows that

$$\frac{dF(x)}{dx} = f(x)$$

wherever this derivative exists.

- F is non-decreasing function, $F(-\infty) = 0$ and $F(\infty) = 1$.

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Mean of a continuous density:

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

$$\text{and } E[H(X)] = \int_{-\infty}^{\infty} h(x) f(x) dx.$$

kth moment:

$$M_k = E(X^k) = \int_{-\infty}^{\infty} x^k f(x) dx.$$

MGF:

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

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Variance of a continuous density

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$
$$= \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$$
$$= E(X^2) - \mu^2.$$

$\boldsymbol{\sigma}$ is referred to as the **standard deviation**

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Example 1: If the probability density of a random variable is given by

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 \le x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

find the probabilities that a random variable having this probability density will take on a value

(a) between 0.2 and 0.8;

(b) between 0.6 and 1.2.

Solution:

$$\text{(a) } P(0.2 \le X \le 0.8) = \int_{0.2}^{0.8} f(x)dx = \int_{0.2}^{0.8} xdx = \left.\frac{x^2}{2}\right|_{0.2}^{0.8} = 0.30.$$

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$$(b)\ P(0.6 \le X \le 1.2) = \int_{0.6}^{1.2} f(x)dx = \int_{0.6}^{1.0} xdx + \int_{1.0}^{1.2} (2-x)dx$$

$$= \frac{x^2}{2}\Bigg|_{0.6}^{1.0} + \left(2x - \frac{x^2}{2}\right)\Bigg|_{1.0}^{1.2} = 0.50$$

Example 2: With reference to the preceding example, find the corresponding distribution function and use it to determine the probabilities that a random variable having this distribution function will take on a value

(a) greater than 1.8;

(b) between 0.4 and 1.6.

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Solution:

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

If $x \le 0$, $F(x) = 0.$

If $0 < x < 1$, $F(x) = \int_0^x t\,dt = \frac{x^2}{2}$

If $1 \le x < 2$, $F(x) = \int_0^1 t\,dt + \int_1^x (2-t)dt = 2x - \frac{x^2}{2} - 1$

If $x \ge 2$, $F(x) = \int_0^1 t\,dt + \int_1^2 (2-t)dt = 1.$

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$$F(x) = \begin{cases} 0 & \text{for } x \le 0 \\ \dfrac{x^2}{2} & \text{for } 0 < x < 1 \\ 2x - \dfrac{x^2}{2} - 1 & \text{for } 1 \le x < 2 \\ 1 & \text{for } x \ge 2 \end{cases}$$

(a) $P(X > 1.8) = 1 - F(1.8) = .02$

(b) $P(0.4 \le X \le 1.6) = F(1.6) - F(0.4) = 0.84.$

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Example 3: Find μ and σ^2 for the probability density of previous example.

Solution:

$$\mu = \int_{-\infty}^{\infty} x f(x)dx = \int_0^1 x^2 dx + \int_1^2 x(2-x)dx = 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^3 dx + \int_1^2 x^2(2-x)dx = \frac{7}{6}$$

$$\sigma^2 = \frac{7}{6} - 1 = \frac{1}{6}.$$