Overview
Chapter 1 & 2

“Ch 1 Must-do-problems”:

17, 20

“Ch 2 Must-do-problems”:

4, 11,14, 17, 19, 35, 36

Learning Objectives

- Signatures of a particle in motion and their mathematical representations

  Velocity, Acceleration, Kinematics

- Choice of coordinate systems that would ease out a calculation.

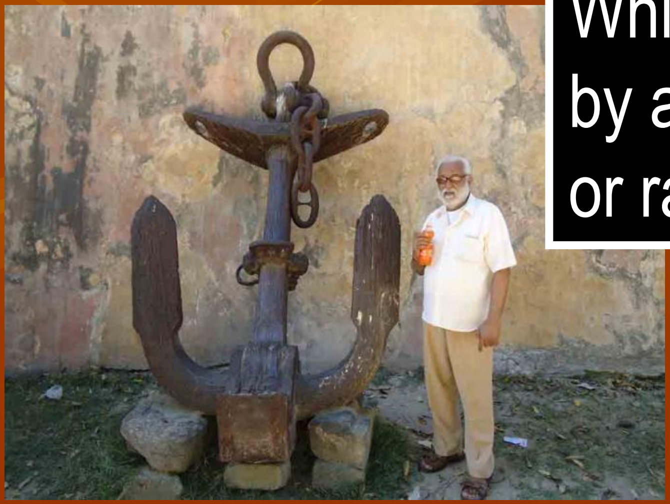
  Cartesian and (Plane) Polar coordinate

- Importance of equation of constraints

  Some application problems

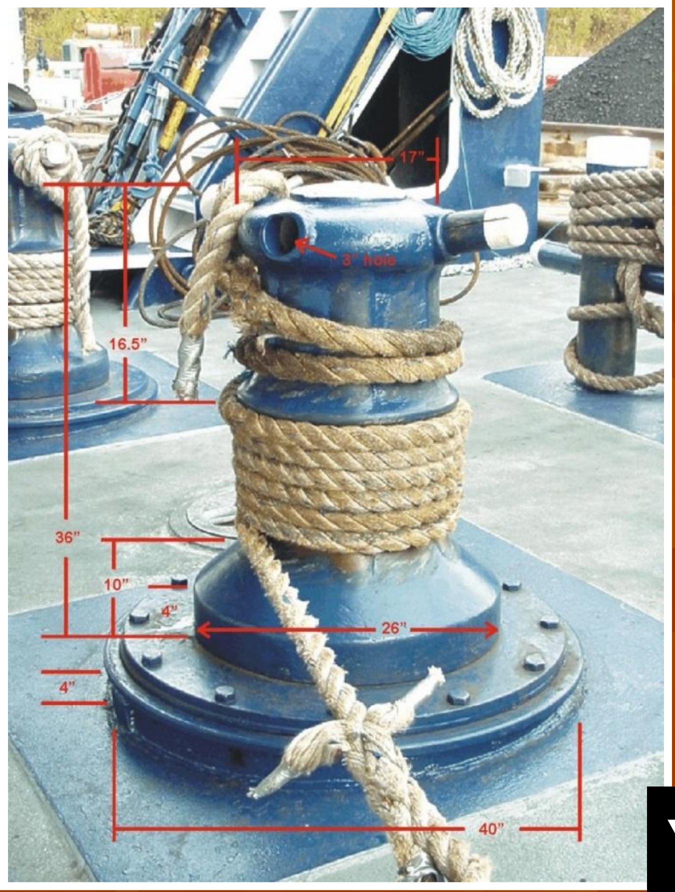
Skip

- Vectors and free body diagram
- Tension in massless string and with strings having mass, string-pulley system.
- Capstan problem



While sailing it is used by a sailor for lowering or raising of an anchor

Capstan



A rope is wrapped around a fixed drum, usually for several turns.

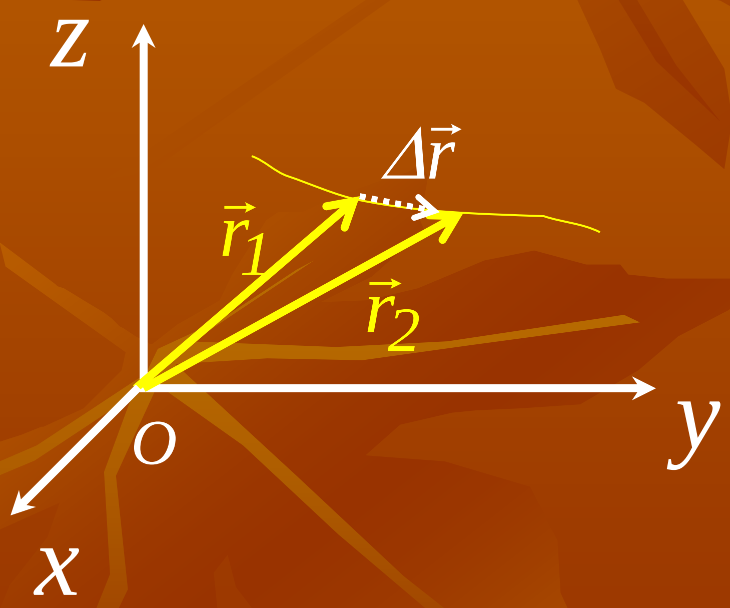
A large load on one end is held with a much smaller force at the other where friction between drum and rope plays an important role

You may look at:

Example 2.13 & Exercise 2.24

Velocity

Particle in translational motion



$$\vec{v} = \lim_{\Delta t \to 0} \frac{\vec{r}_2 - \vec{r}_1}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t}$$

$$= \frac{d\vec{r}}{dt} = \dot{\vec{r}}$$

"r-naught"

Position vector in Cartesian system $\Rightarrow \vec{r}_{(xyz)} = \hat{i}x + \hat{j}y + \hat{k}z$

Acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}}$$

Kinematics in Genaral

In our prescription of "r-naught", Newton's 2nd law:

$$m\ddot{\vec{r}} = \vec{F}\left(\vec{r},\dot{\vec{r}},t\right)$$

To know precisely the particle trajectory, We need:

- ✓ Information of the force
- ✓ Two-step integration of the above (diff) equation
- ✓ Evaluation of constants and so initial conditions must be supplied
- ✓ A system of coordinate to express position vector

Simple form of Forces

- $\vec{F} = m\vec{a}, \quad \vec{a} \rightarrow constant$
  (Force experienced near earth's surface)
- $\vec{F} = \vec{F}(t)$
  (Effect of radio wave on Ionospheric electron)
- $\vec{F} = \vec{F}(\vec{r})$ (Gravitational, Coulombic, SHM)
- $\vec{F} = \vec{F}(\dot{\vec{r}})$ (Any guess ???)

Example 1.11

Find the motion of an electron of charge e and mass m, which is initially at rest and suddenly subjected to oscillating electric field, $\vec{E} = \vec{E}_0 \sin\omega t$.

We check:

- Information of force.
- Initial condition.

We decide:

The coordinate system to work on

Solution

Translate given information to mathematical form:

1. $m\ddot{\vec{r}} = -\vec{E}e \Rightarrow \ddot{\vec{r}} - \dfrac{\vec{E}e}{m} = -\dfrac{e\vec{E}_0 \sin \omega t}{m}$

As E_0 is a constant vector, <u>the motion is safely 1D</u>

$$\ddot{x} = -\frac{eE_0 \sin \omega t}{m} = a_0 \sin \omega t \qquad \boxed{a_0 = -\frac{eE_0}{m}}$$

…..but the acceleration is **NOT** constant

Solution Contd.

Step 1: Do the first integration

$$\dot{x} = -\frac{a_0}{\omega}\cos\omega t + C_1$$

Step 2: Plug in the initial conditions to get C_1

At, t = 0, $\dot{x} = 0 \Rightarrow c_1 = \frac{a_0}{\omega}$

Hence, $$\boxed{\dot{x} = -\frac{a_0}{\omega}\cos\omega t + \frac{a_0}{\omega}}$$

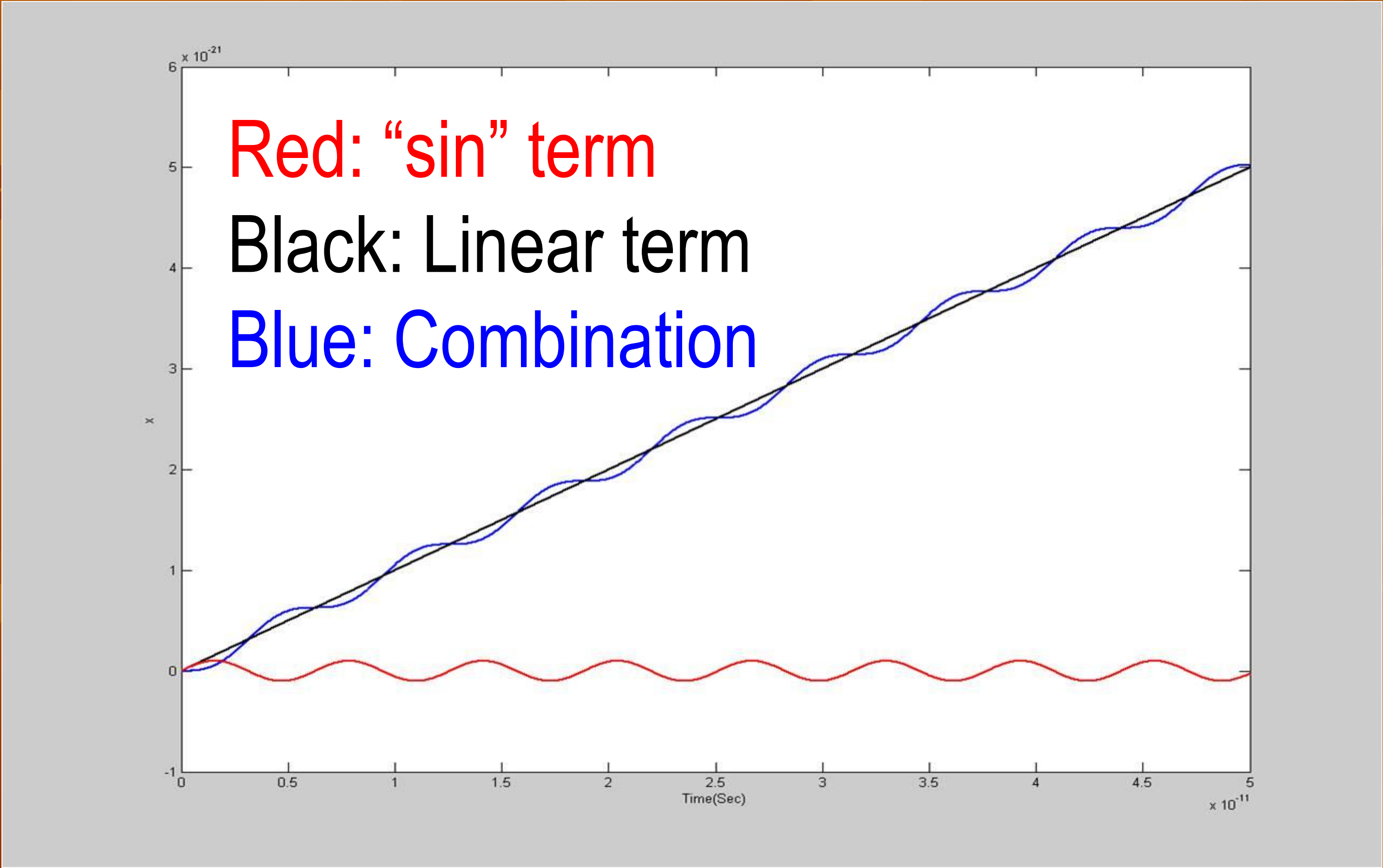
Solution Contd.

Step 3: Carry out the 2nd. Integration and get the constant thro' initial conditions

$$x = \frac{a_0}{\omega} t - \frac{a_0}{\omega^2} \sin \omega t$$

What would be the fate of the electrons?

Graphical Representation



Viscous Drag

- Motion of a particle subjected to a resistive force

Examine a particle motion falling under gravity near earth's surface taking the frictional force of air proportional to the first power of velocity of the particle. The particle is dropped from the rest.

Solution

Equation of motion:

$$F = m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = mg - Cv$$

Resistive force proportional to velocity (linear drag force)

Final Equation

Solving the differential equation and plucking the correct boundary condition (Home Exercise)

$$v = \frac{dx}{dt} = \frac{g}{k}\left(1 - e^{-kt}\right)$$

$$k = \frac{C}{m}$$

If the path is long enough, $t \to \infty$

$$v = \frac{g}{k} = v_{ter}$$

"Terminal Velocity"

Important Results & A Question

Some calculated values of terminal velocity is given:

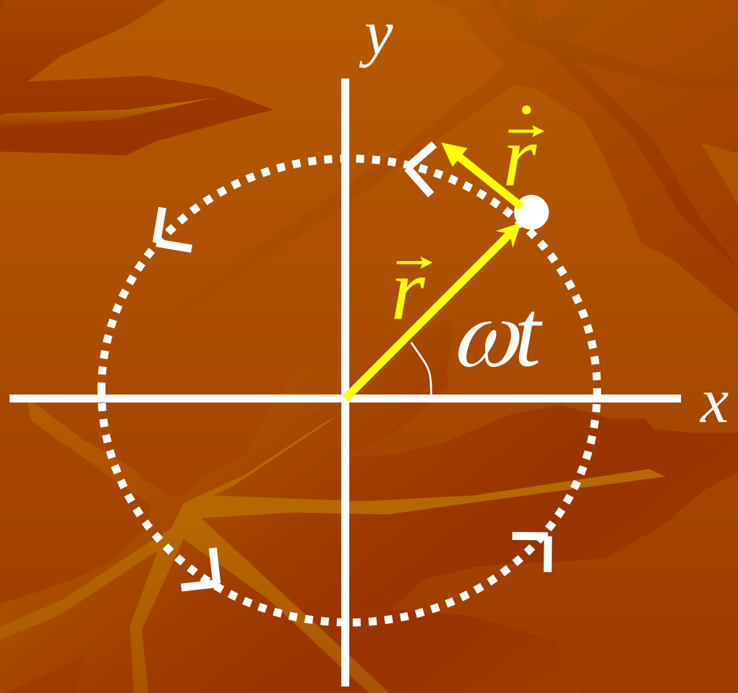
1. Terminal velocity of spherical oil drop of diameter 1.5 μm and density 840kg/m^3 is 6.1x10^{-5} m/s. This order of velocity was just ok for Millikan to record it thro' optical microscope, available at that time.

2. Terminal velocity of fine drizzle of dia 0.2mm is approximately 1.3m/s.

How one would estimate these values?

Circular Motion

Particle in uniform circular motion:



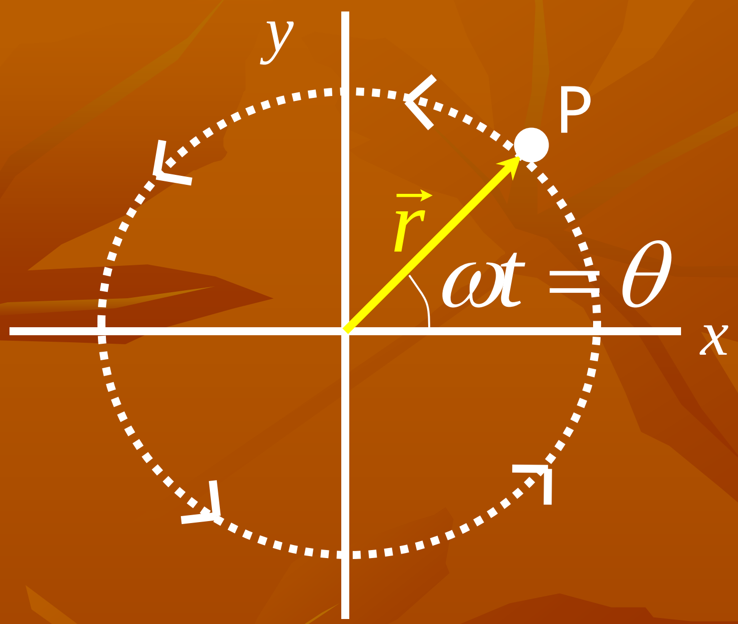
$$\vec{r} = \hat{i}x + \hat{j}y$$
$$= r\left(\hat{i}\cos\omega t + \hat{j}\sin\omega t\right)$$

$$\dot{\vec{r}} = \vec{v}$$
$$= r\omega\left(-\hat{i}\sin\omega t + \hat{j}\cos\omega t\right)$$

$$\Rightarrow \dot{\vec{r}}.\vec{r} = 0$$

...as expected

Coordinate to Play



$$\vec{r}_{at\,P} = \hat{i}x + \hat{j}y$$

Position vector in Cartesian form where we need to handle two unknowns, x and y

But also, $$\vec{r}_{at\,P} = \vec{r}(r, \theta)$$

Position vector in Polar form

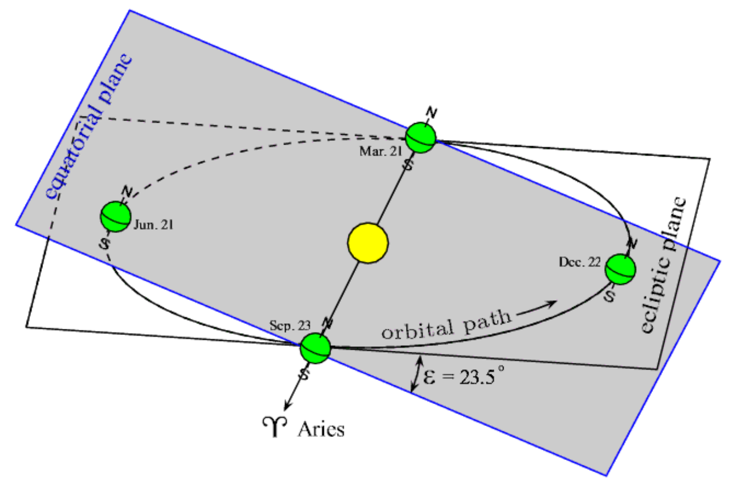
The Key:

For this case, r = const & particle position is completely defined by just one unknown θ

Coordinate Systems

Choice of coordinate system is dictated (mostly) by symmetry of the problem

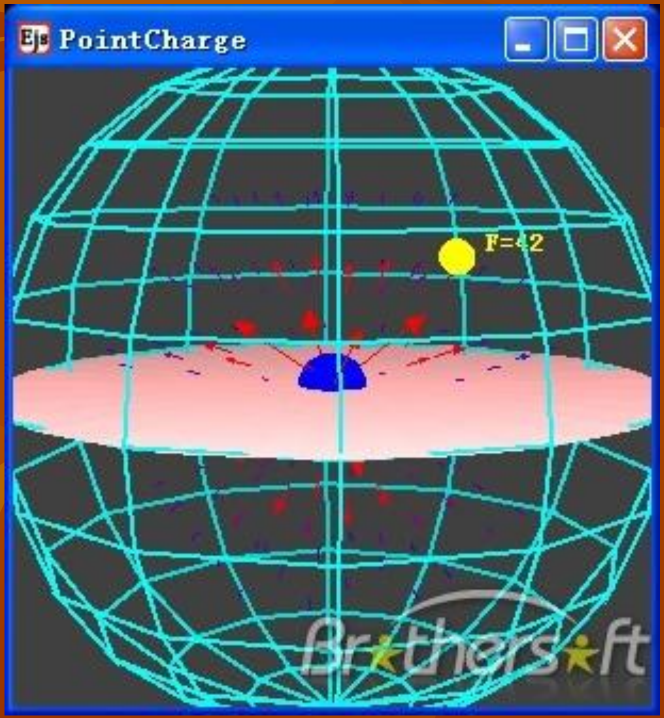
- Motion of earth round the sun

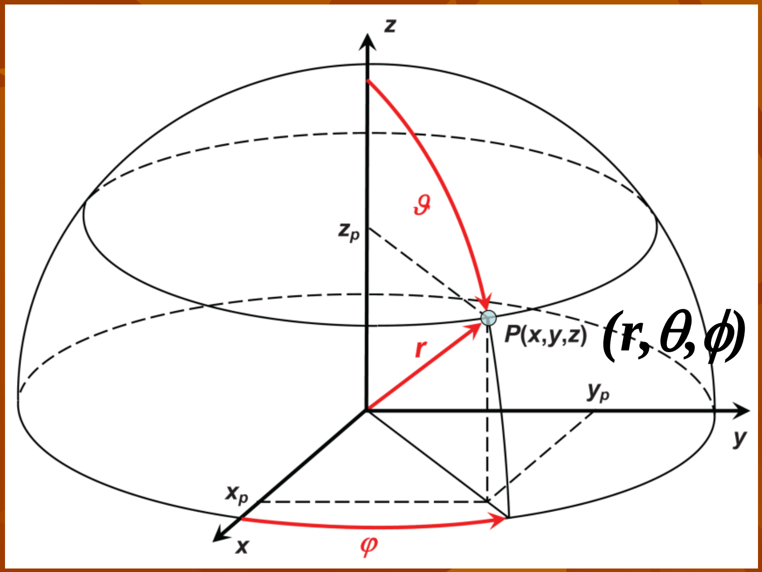


Plane polar coordinate (r, θ)

Other Examples

- Electric Field around a point charge

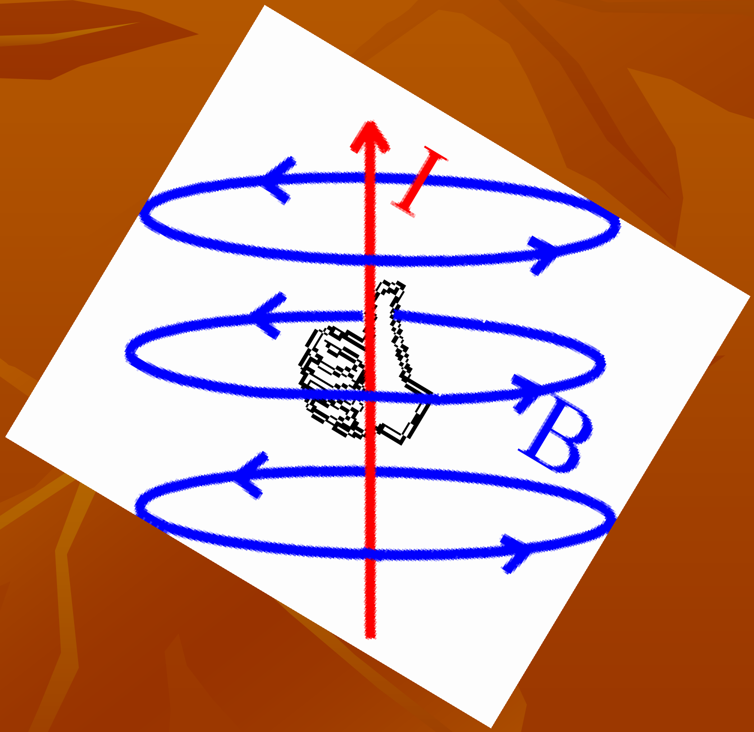


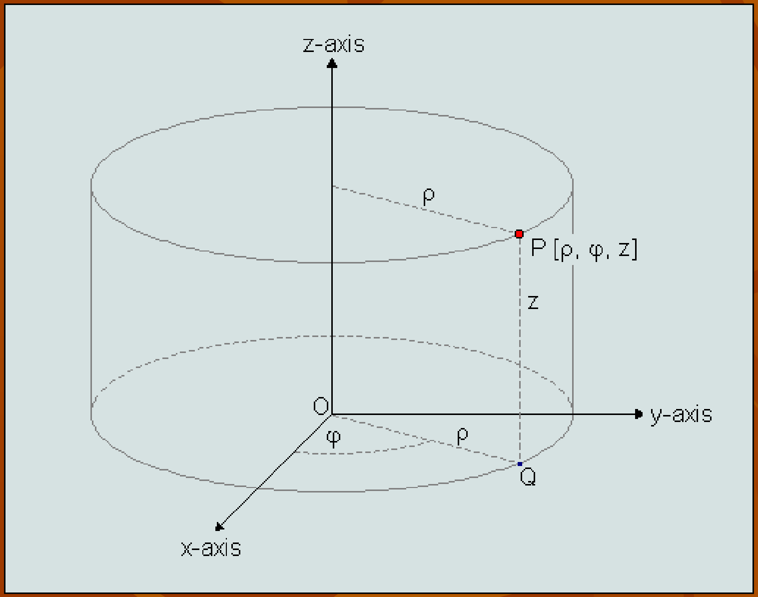


Spherical polar coordinate P (r, θ, ϕ)

Other Examples

- Mag. field around a current carrying wire

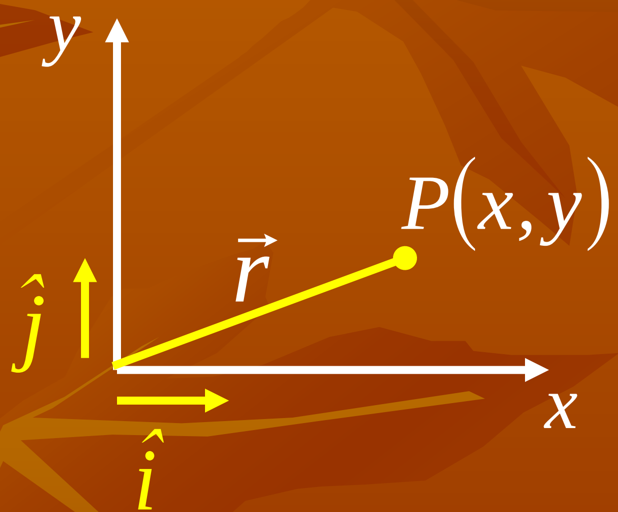


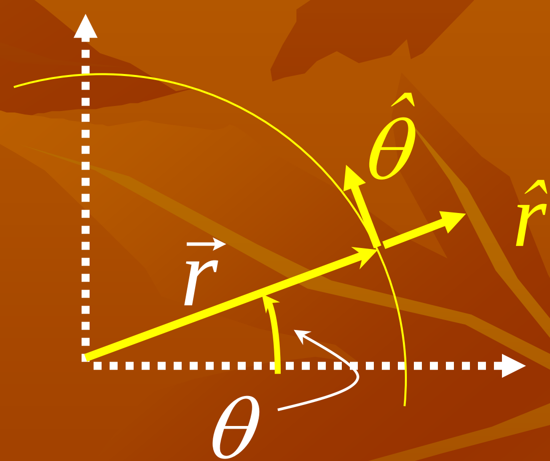


Cylindrical polar coordinate P (ρ, ϕ, z)

Plane Polar System

How to develop a plane polar coordinate system ?

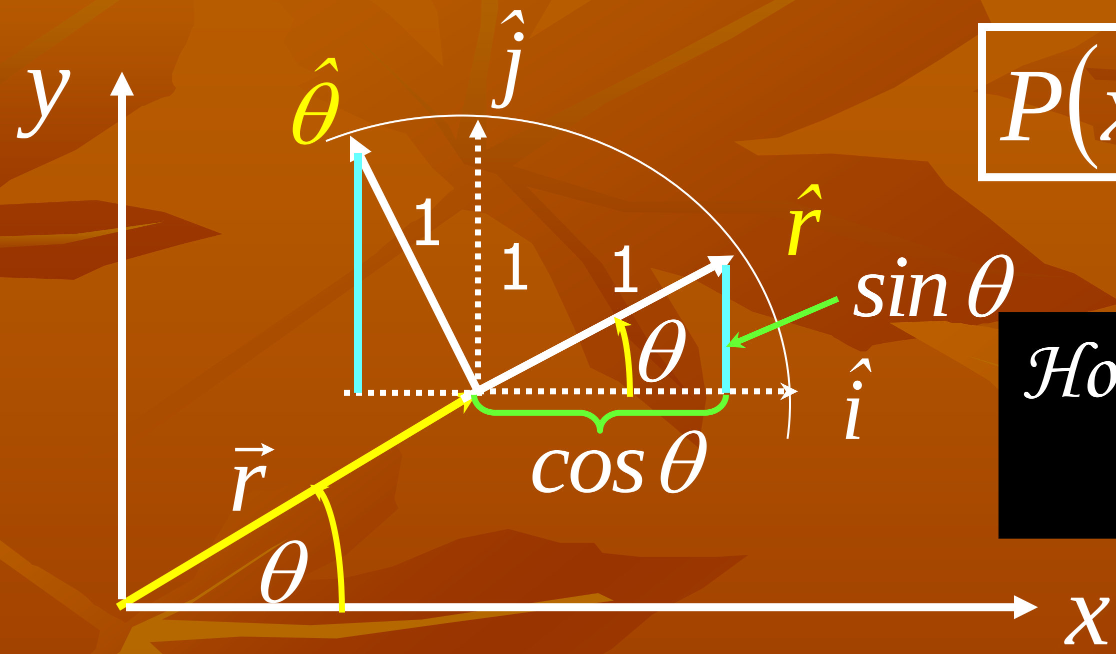




<u>To Note:</u>

Unit vectors are drawn in the direction of increasing coordinate….**THE KEY**

Relation with (XY)



$$P(x,y) \equiv \vec{r} = \hat{i}x + \hat{j}y$$

How pt P (x,y) expressed in polar form??

$$\vec{r} = \hat{r}r$$

$$\hat{r} = \hat{i}\cos\theta + \hat{j}\sin\theta$$

$$\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta$$

Q. Is θ missing here?

Unit Vectors

$$\hat{r} = \hat{i}\cos\theta + \hat{j}\sin\theta$$

$$\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta$$

1. Similarity: Use of Cartesian & Polar unit vectors to express a vector and are **orthogonal** to each other

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

$$\hat{i}.\hat{j} = 0 \Rightarrow \hat{r}.\hat{\theta} = 0$$

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta}$$

2. Dissimilarity: ???????.

Polar unit vectors are **NOT** constant vectors; they have an inherent θ dependency…. **(REMEMBER)**

Consequence

As polar unit vectors are not constant vectors, it is meaningful to have their time derivatives

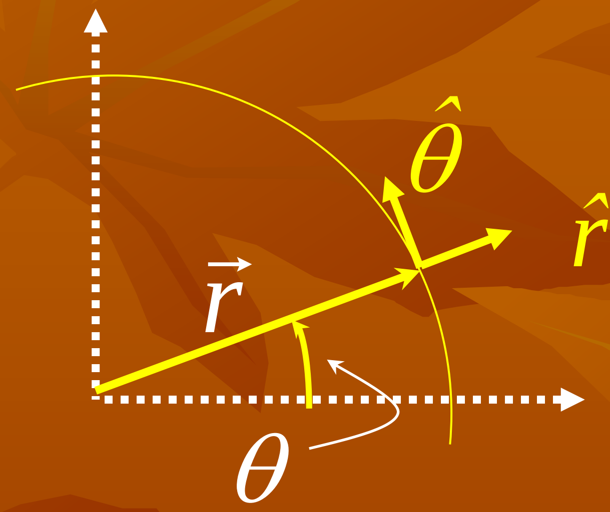
⇓

Useful expressions of velocity & acceleration

$$\vec{r} = \hat{i}x + \hat{j}y \quad \overset{velocity}{\Longrightarrow} \quad \dot{\vec{r}} = \hat{i}\dot{x} + \hat{j}\dot{y}$$

$$\vec{r} = r\hat{r} \quad \Longrightarrow \quad ???$$

Velocity Expression



$$\vec{v} = \dot{\vec{r}} = \frac{d}{dt}\left(r\hat{r}\right) = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Conclusion

$$\vec{v} = \dot{r}\hat{r} + \underline{r\dot{\theta}}\hat{\theta}$$

"Radial Part v_r" "Tangential Part v_θ"

In general, the **velocity expressed in plane polar** coordinate has a **radial** and a **tangential** part. For a **uniform circular** motion, it is **purely tangential**.

Acceleration

$$\vec{a} = \ddot{\vec{r}} = \frac{d\dot{\vec{r}}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right)$$

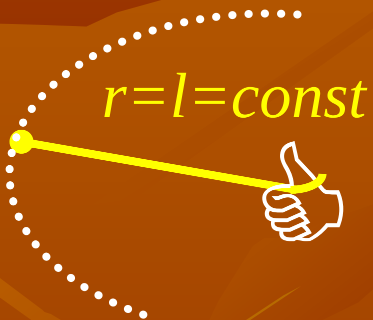
Do these steps as HW

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}$$

"Radial Part a_r" "Tangential Part a_θ"

Example

Find the acceleration of a stone twirling on the end of a string of fixed length using its polar form



$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}$$

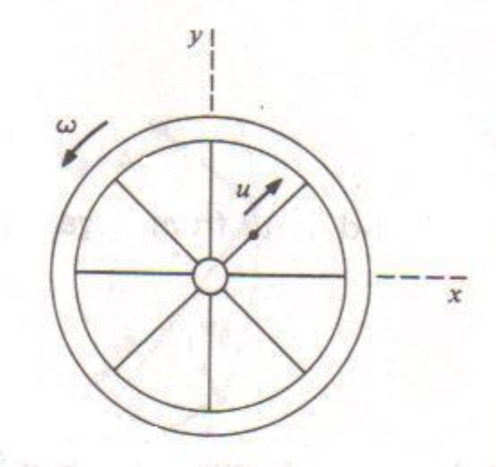
$$\vec{a} = \left(-r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta}\right)\hat{\theta}$$

$$= -\underline{r\omega^2}\,\hat{r} + \underline{r\alpha}\,\hat{\theta}$$

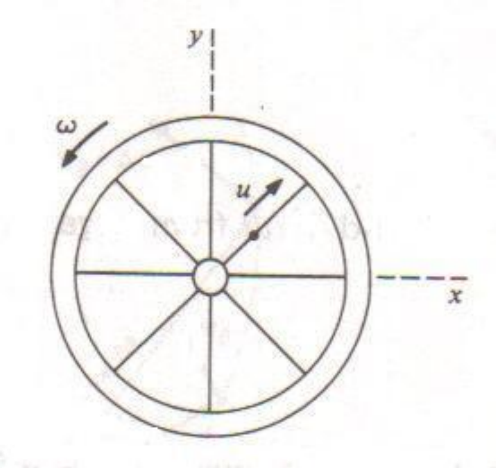
"Centripetal" accln. (v^2/r)   Tangential accln.

Example 1.14

A bead moves along the spoke of a wheel at a constant speed u mtrs per sec. The wheel rotates with uniform angular velocity $\dot{\theta} = \omega$ rad. per sec. about an axis fixed in space. At $t = 0$, the spoke is along x-axis and the bead is at the origin. Find vel and accl. in plane polar coordinate



Solution (Velocity)



$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$r = ut \qquad \dot{r} = u; \qquad \dot{\theta} = \omega$$

$$\vec{v} = u\hat{r} + ut\omega\hat{\theta}$$

Solution (Acceleration)

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}$$

$$r = ut; \qquad \dot{r} = u; \qquad \ddot{r} = 0$$

$$\dot{\theta} = \omega \qquad \ddot{\theta} = 0$$

$$\vec{a} = \left(-ut\omega^2\right)\hat{r} + \left(2u\omega\right)\hat{\theta}$$

Newton's Law (Polar)

Q. How Newton's Law of motion look like in polar form?

In 2D Cartesian form (x,y) :

$$F_x = m\ddot{x} \qquad F_y = m\ddot{y}$$

In 2D Polar form (r,θ), can we write:

$$F_r = m\ddot{r} \qquad F_\theta = m\ddot{\theta}$$

?

Newton's Law in Polar Form

Acceleration in polar coordinate:

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}$$

$$= a_r\hat{r} + a_\theta\hat{\theta}$$

$$F_r = m\left(\ddot{r} - r\dot{\theta}^2\right) \neq m\ddot{r}$$

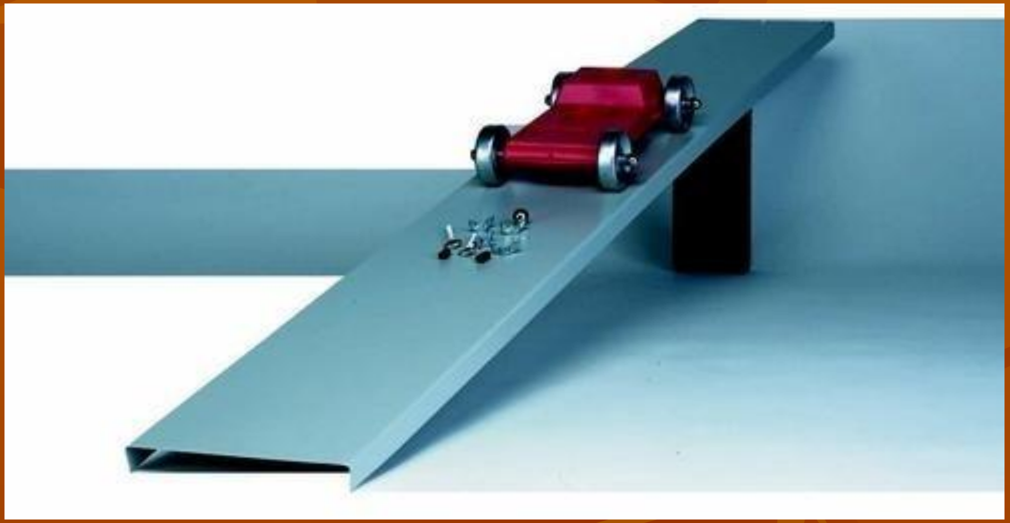
$$F_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) \neq m\ddot{\theta}$$

Newton's eq. in polar form

Newton's eq.s in polar form **DO NOT** scale in the same manner as the Cartesian form

Constraint

Any restriction on the motion of a body is a constraint

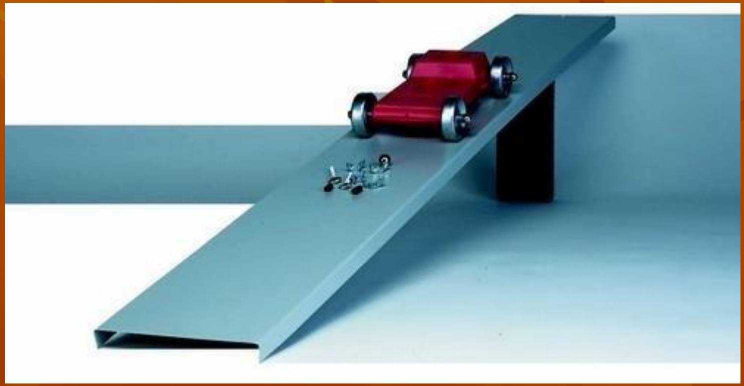


How these things are associated with constraints?

More on Constraints

⇒ Bob constrained to move on an arc of a circle and tension T in the string is the force of constraint associated with it

The rollers roll down the incline without flying off and the normal reaction is the force of constraint associated \with it ⇐



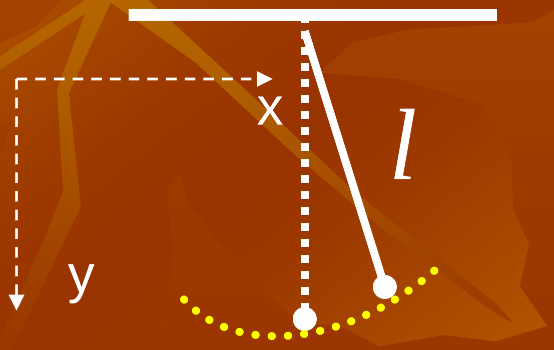
Common → Both are rigid bodies where rigidity is the constraint and force associated with it is ....

Example 1

Constraints can be viewed mathematically by one/more **eq. of constraint** in the coordinate system it is used to express

(Note: Constraints may involve inequations too)

Take a simple pendulum with massless string



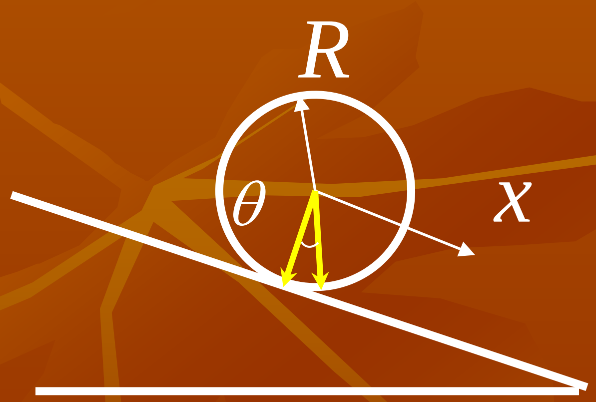
$$x^2 + y^2 = l^2 = const.$$

$$z = 0$$

Eq.s of constraint

Example 2

If a drum of radius R has to roll down a slope without slipping, what is the corresponding eq. of constraint ?



Eq. of constraint:

$$R\theta = x \Rightarrow R\ddot{\theta} = \ddot{x}$$

Q. How eq. of constraints help us get **useful information**?

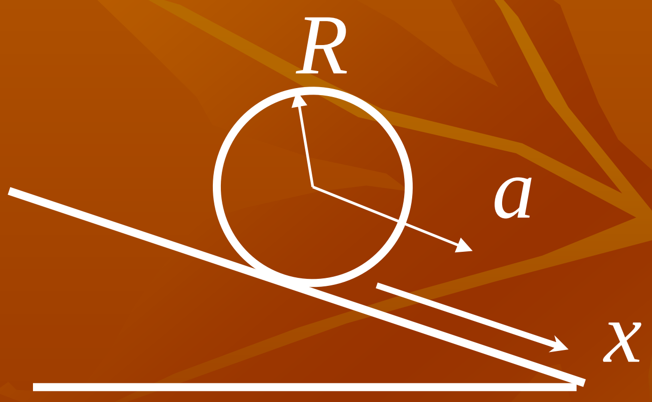
Useful Information

For a drum that is rolling without slipping down an incline from rest, its angular acceleration (α) is readily found using the equation of constraint.

$$R\theta = x = \frac{1}{2}at^2$$

$$\alpha = \ddot{\theta} = \frac{a}{R}$$

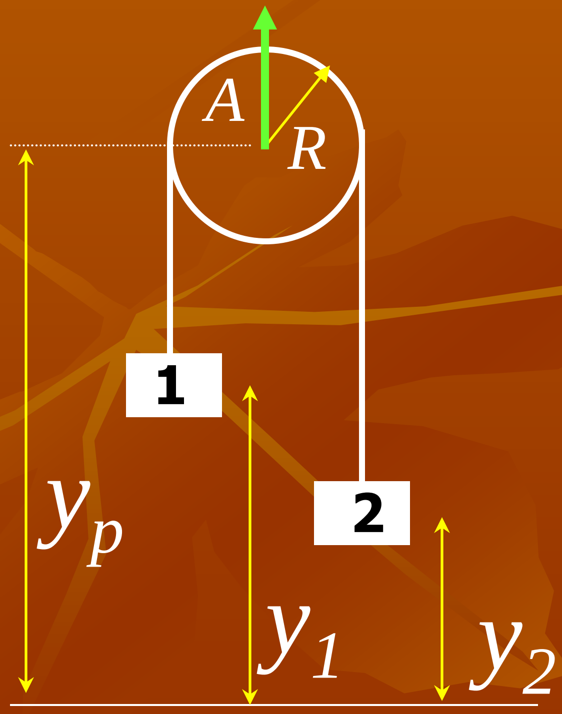
(Prob 1.14)



Home Study: **Ex. 2.4a**

Ex 2.4b

A pulley accelerating upward at a rate *A* in a "Mass-Pulley" system. Find how acceleration of bodies are related (rope/pulley mass-less).

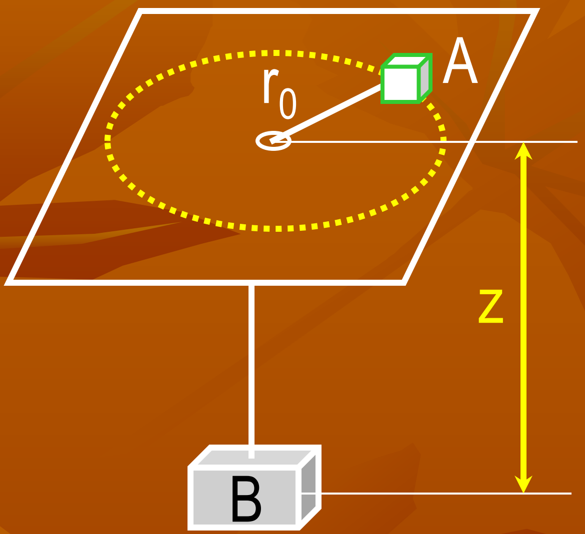


Constrain Equation:

$$l = \pi R + \left(y_p - y_1\right) + \left(y_p - y_2\right)$$

$$\ddot{y}_p = \frac{1}{2}\left(\ddot{y}_1 + \ddot{y}_2\right)$$

Ex 2.7



Initial conditions:

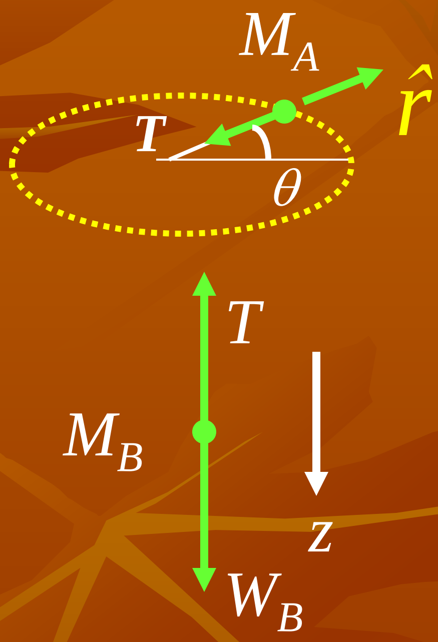
- B stationary and A rotating by a mass-less string with radius r_0 and constant angular velocity ω_0. String length = l

Q. If B is released at t=0, what is its acceleration immediately afterward?

Steps:

1. Get the information of force
2. Write the eq. of motions
3. Get the constraint eq. if there is any and use it effectively

Ex 2.7 contd.



Eq.s of motion:

$$M_A\left(\ddot{r} - r\dot{\theta}^2\right) = -T \qquad (1)$$

$$M_A\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0 \qquad (2)$$

$$M_B\ddot{z} = W_B - T \qquad (3)$$

And, the eq. of constraint:

$$r + z = l \Rightarrow \ddot{r} = -\ddot{z} \qquad (4)$$

Ex 2.7 contd.

Using the eq.s of motion and the constraint:

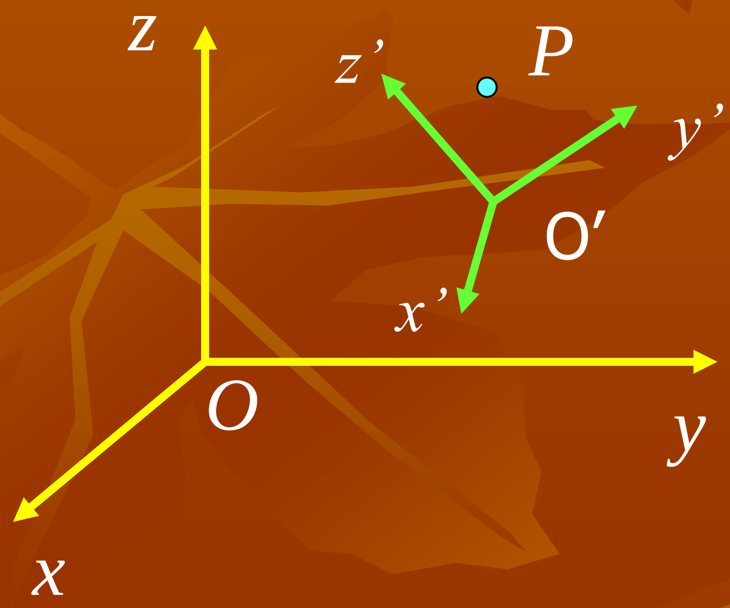
$$\ddot{z}(t) = \frac{W_B - M_A r \dot{\theta}^2}{M_A + M_B}$$

⬇

$$\ddot{z}(0) = \frac{W_B - M_A r_0 \omega_0^{\,2}}{M_A + M_B}$$

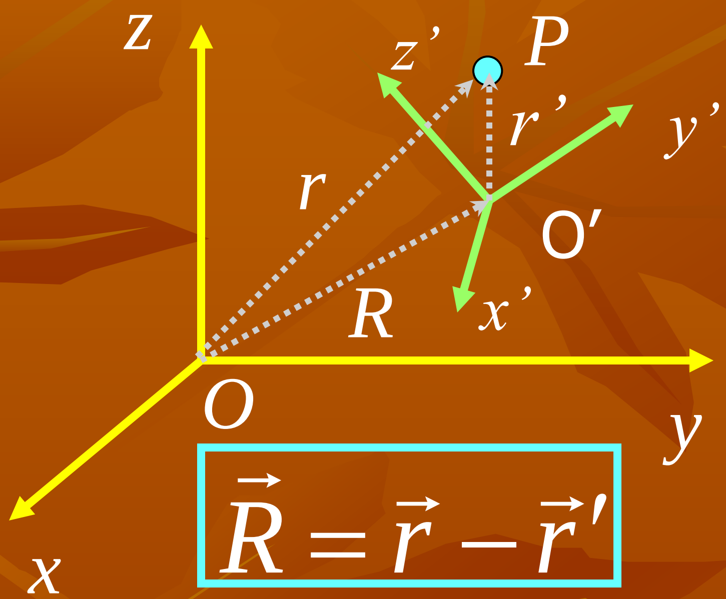
Fictitious Force

Two observers O and O’, fixed relative to two coordinate systems Oxyz and O’x’y’z’, respectively, are observing the motion of a particle P in space.



Find the condition when to both the observers the particle appears to have the same force acting on it?

Formulation



$$\vec{R} = \vec{r} - \vec{r}'$$

“Inertial Frames”

We look for:

$$\vec{F} = \vec{F}'$$

$$m\frac{d^2\vec{r}}{dt^2} = m\frac{d^2\vec{r}'}{dt^2}$$

$$\frac{d^2(\vec{r} - \vec{r}')}{dt^2} = \frac{d^2\vec{R}}{dt^2} = 0$$

$$\frac{d\vec{R}}{dt} = Constant$$

Non-inertial Frames

$$If\ O'x'y'z'\ is\ accelerating \Rightarrow \frac{d\vec{R}}{dt} \neq Constant$$

To observers O and O', particle will have different forces acting on it

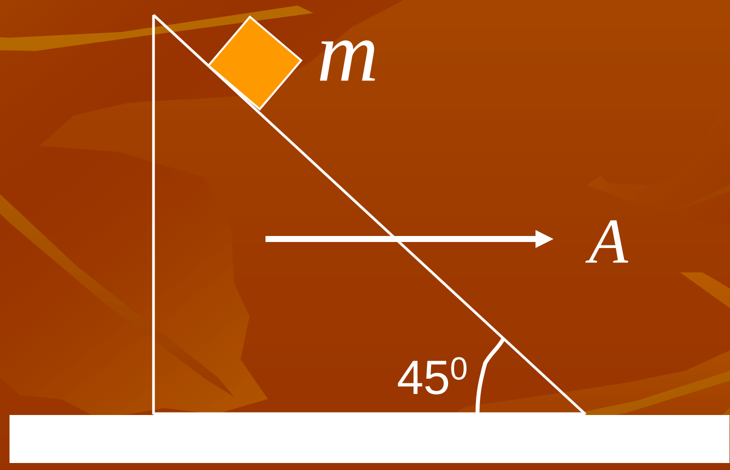
$$m\frac{d^2\vec{r}'}{dt^2} = m\frac{d^2\vec{r}}{dt^2} - m\frac{d^2\vec{R}}{dt^2}$$

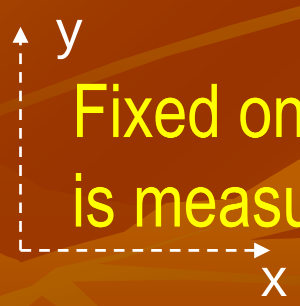
$$\vec{F}' = \vec{F}_{true} + \vec{F}_{fictitious}$$

Problem

Exercise 2.16

A 45^0 wedge is pushed along a table with a constant acceleration *A*. A block of mass *m* slides without friction on the wedge. Find the acceleration of the mass m.

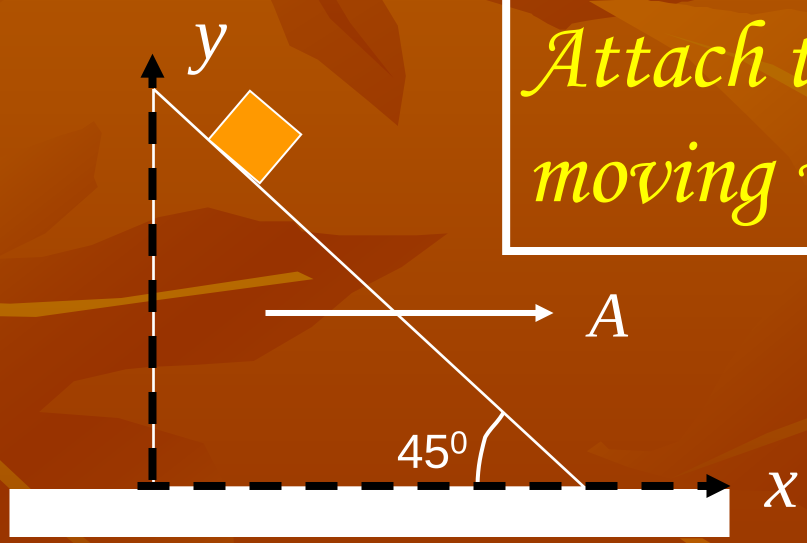




Fixed on Table and "*A*" is measured here

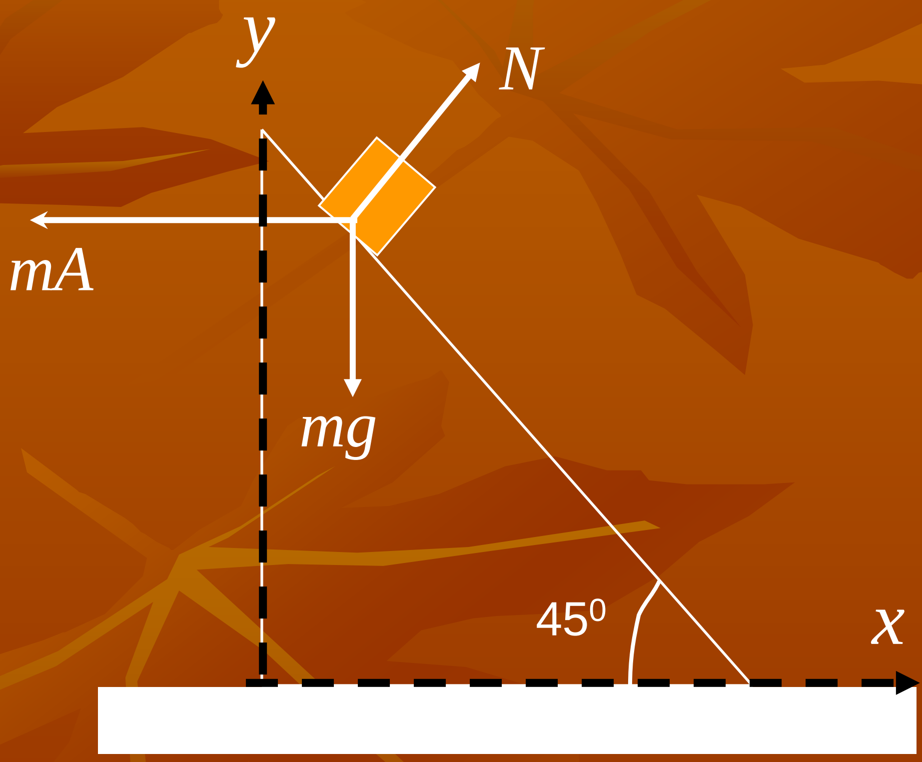
Solution

We would like to use the concept of fictitious force.....



Attach the ref. frame to the moving wedge....

Force Diagram



$$N = \frac{mg}{\sqrt{2}} + \frac{mA}{\sqrt{2}}$$

$$f_{surface} = \frac{mg}{\sqrt{2}} - \frac{mA}{\sqrt{2}}$$

Accelerations

Acceleration of the block down the incline:

$$a_{surface} = \frac{g}{\sqrt{2}} - \frac{A}{\sqrt{2}}$$

$$a_x = \left(\frac{g}{\sqrt{2}} - \frac{A}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}$$

$$a_y = -\left(\frac{g}{\sqrt{2}} - \frac{A}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}$$

These are wrt the wedge!

"Real" Accelerations

To find the acceleration of m, we must recollect:

$$m\frac{d^2\vec{r}'}{dt^2} = m\frac{d^2\vec{r}}{dt^2} - m\frac{d^2\vec{R}}{dt^2}$$

$$\Rightarrow \vec{a}' = \vec{a}_{true} + \vec{a}_{fictitious}$$

$$a_{x'} = \left(\frac{g}{\sqrt{2}} - \frac{A}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} + A$$

$$= \frac{g}{2} + \frac{A}{2}$$

$$a_{y'} = \left(\frac{A}{2} - \frac{g}{2}\right)$$